

RINGS OF POWER OPERATIONS FOR MORAVA E -THEORIES ARE KOSZUL

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ABSTRACT. We show that the ring of power operations for any Morava E -theory is Koszul.

1. INTRODUCTION

Given a structured commutative ring spectrum E , it is an important task to understand its theory of *power operations*. For this paper, power operations are *additive* operations on the homotopy groups of commutative E -algebras which arise as the residue of the (non-linear) multiplicative structure of a structured commutative ring spectrum. The closest classical analogue to power operations is the Frobenius map on commutative rings of finite characteristic; indeed there is a close connection between power operations and the Frobenius map. The most familiar examples of power operations are the Steenrod operations on the mod p homology of a space, which are in fact a specialization of more general operations defined on the homotopy of a commutative $H\mathbb{F}_p$ -algebra, where $H\mathbb{F}_p$ denotes the mod p Eilenberg-Mac Lane spectrum.

The notion of Koszul algebra was introduced by Priddy [Pri70]. Roughly speaking, a Koszul algebra is a graded k -algebra A which admits a *Koszul complex*, namely a functorial resolution $\cdots \rightarrow C_n \otimes_k M \rightarrow C_{n-1} \otimes_k M \rightarrow \cdots$ of right A -modules M , which is “minimal”, in the sense that the Koszul complex for $M = k$ has trivial differentials.

In this paper, we address the ring of power operations for a Morava E -theory spectrum. Thus, we fix a Morava E -theory spectrum associated to the universal deformation of a formal group of height n at a prime p . By a theorem of Hopkins and Miller, E is a commutative S -algebra, and therefore we may consider the category of commutative E -algebra spectra Alg_E . Each Morava E -theory comes with a ring Γ of power operations; elements of Γ are precisely the natural endomorphisms of the functor $\pi_0 L_{K(n)}: h\text{Alg}_E \rightarrow \text{Ab}$. (Here $L_{K(n)}$ is the $K(n)$ -**localization** functor; many of the E -algebras of interest (including E itself) satisfy $A \approx L_{K(n)}A$, in which case elements of Γ give endomorphisms of $\pi_0 A$.) A precise definition of Γ is given in §3.

Main Theorem. *The ring Γ of power operations for any Morava E -theory is Koszul.*

A subtlety is that although Γ contains the coefficient ring $E_0 = \pi_0 E$, this subring is not central in Γ . The notion of Koszul we will use (described in §4) will make sense for such rings. Furthermore, with this definition, it will be a *consequence* of the main theorem that the ring Γ is *quadratic*, i.e., admitting a presentation in which relations are quadratic in the generators.

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1.1. Sketch of the proof. We briefly indicate here the structure of the proof. We fix a Morava E -theory spectrum E . The argument will not apply directly to the ring Γ of additive power operations for E , but rather to the ring Δ of operations which act on the “cotangent space” of the homotopy of an augmented E -algebra spectrum. In fact, $\Delta \approx \Gamma$ as graded rings (3.16), so it suffices to show that Δ is Koszul.

Consider the functor \tilde{C} (2.5) which associates to a space X the free non-unital E_∞ -algebra on X :

$$\tilde{C}(X) \approx \coprod_{m \geq 1} X_{h\Sigma_m}^m.$$

There is an analogous functor \tilde{D} (2.6) on spectra, given by

$$\tilde{D}(Y) \approx \bigvee_{m \geq 1} Y_{h\Sigma_m}^{\wedge m},$$

and we have that $\Sigma_+^\infty \tilde{C} \approx \tilde{D} \Sigma_+^\infty$.

In general, given a functor F from an additive category to an abelian category, we can define a linearization (5.1) of F , by

$$\mathcal{L}_F(X) \stackrel{\text{def}}{=} \text{Cok} [F(p_1 + p_2) - F(p_1) - F(p_2): F(X \oplus X) \rightarrow F(X)],$$

where $p_1, p_2: X \oplus X \rightarrow X$ are the two projections.

Let $E_*^\wedge X$ denote the completed E -homology (3.1) of a space X . Applying linearization to the composite functor $F = E_*^\wedge D$, with $X = S^0$, we are lead to consider the cokernel of a map

$$E_*^\wedge D(S^0 \vee S^0) \rightarrow E_*^\wedge D(S^0)$$

The first step (5.7) of the proof is to identify this cokernel (the linearization of F at S^0) with the underlying E_* -module of an algebra Δ of power operations (see §3). There is a decomposition $\Delta \approx \bigoplus_{k \geq 0} \Delta[k]$, where $\Delta[k]$ comes from the $m = p^k$ summand in C . (We actually state and prove this step in terms of an algebraic approximation functor $\tilde{\mathbb{T}}$ (§3.2), which has the property that $\tilde{\mathbb{T}}(\bigoplus E_*) \approx E_*^\wedge \tilde{D}(\bigvee S^0)$.)

Because \tilde{D} is a monad on the homotopy category of spectra, we may consider the two-sided bar construction $\mathcal{B}(\tilde{D}) = \mathcal{B}(\tilde{D}, \tilde{D}, \tilde{D})$. A similar argument (5.8) identifies the cokernel of the analogous map

$$E_*^\wedge \mathcal{B}(\tilde{D})(S^0 \vee S^0) \rightarrow E_*^\wedge \mathcal{B}(\tilde{D})(S^0)$$

(i.e., the linearization of $F = E_*^\wedge \mathcal{B}(\tilde{D})$ at S^0) with the two-sided bar construction $\mathcal{B}(\Delta) = \mathcal{B}(\Delta, \Delta, \Delta)$ of the ring Δ . (Again, our actual statement is given in terms of the bar construction $\mathcal{B}(\tilde{\mathbb{T}})$ of the algebraic approximation functor $\tilde{\mathbb{T}}$.)

What we are actually interested in is a certain quotient $\overline{\mathcal{B}}(\Delta)$ of $\mathcal{B}(\Delta)$, which is isomorphic to $\mathcal{B}(E_*, \Delta, E_*)$. This quotient admits (4.1) a decomposition $\bigoplus_{k \geq 0} \overline{\mathcal{B}}(\Delta)[k]$ associated to the decomposition of Δ . We observe (4.4) that Δ is Koszul if and only if the homology of $\overline{\mathcal{B}}(\Delta)[k]$ is concentrated in degree k , for all $k \geq 0$. More precisely, we take this homological vanishing property as the *definition* of being Koszul; the discussion of §4 explains why this is the correct definition. In particular, we show that with this definition, if a ring is *Koszul* then it is necessarily *quadratic* (4.10). This also means that we do not need to first construct

an admissible basis (or any basis at all) for Δ , as is typical in many proofs of the Koszul property.

To prove that $H_*\overline{\mathcal{B}}(\Delta)[k]$ is concentrated in degree k , we look at the combinatorics of the bar construction $\mathcal{B}(\tilde{C}) = \mathcal{B}(\tilde{C}, \tilde{C}, \tilde{C})$, which are governed by partitions. In particular, there is a weak equivalence of simplicial spaces

$$\mathcal{B}(\tilde{C})(X) \approx \coprod_{m \geq 0} (P_m \times X^m)_{h\Sigma_m}$$

where P_m is the nerve of the poset of partitions of an m -element set (6.2). (The simplicial coordinate comes from the simplicial set P_m .) Translating this into a statement about $\overline{\mathcal{B}}(\Delta)$, we discover (7.15) that $\overline{\mathcal{B}}(\Delta)[k]$ is isomorphic to the cokernel of a certain map

$$E_*^\wedge(\overline{P}_m \wedge (S^0 \vee S^0)^{\wedge m})_{h\Sigma_m} \rightarrow E_*^\wedge(\overline{P}_m \wedge (S^0)^{\wedge m})_{h\Sigma_m}$$

where $m = p^k$, and \overline{P}_m is certain quotient of P_m . This cokernel is denoted $\tilde{Q}(\overline{P}_m)$ in the text, where $\tilde{Q}(Y)$ is called the *transitive E -homology* (7.1) of a Σ_m -space Y . That is, $\tilde{Q}(Y)$ is the linearization of the functor $X \mapsto E_*^\wedge(\Sigma_+^\infty Y \wedge X^{\wedge m})_{h\Sigma_m}$ evaluated at $X = S^0$.

Thus, the proof is reduced to showing that the simplicial abelian group $\tilde{Q}(\overline{P}_m)$ has its homology concentrated in degree k . We do this in two steps. First, we define a subcomplex $U_m \subseteq P_m \times (\Sigma_m/\Sigma_p \wr \cdots \wr \Sigma_p)$, which we call the uniform complex (6.9). We show (8.8) that to prove the vanishing of homology for $\tilde{Q}(\overline{P}_m)$, it suffices to prove the analogous result for $\tilde{Q}(\overline{U}_m)$, where \overline{U}_m is a certain quotient of U_m . Second, we prove this vanishing result for \overline{U}_m , by a “shellability” method (9.1), inspired by the Solomon-Tits calculation of the homology of Tits buildings.

Thus, the **proof of the main theorem** is obtained from the combination of (7.15), (8.8), and (9.1), applied to our definition of Koszul from §4.4, and using the isomorphism (3.16).

Our argument has a somewhat formal character. Mostly, it concerns the combinatorics of bar constructions and partition complexes, and one might want to generalize it to more general examples of commutative S -algebras. We note that there are a couple of places where the argument really does use something about Morava E -theory. In particular, at several points we need to know that the ring Δ is projective as an E_* -module; this is a result due to Strickland. We need this in order to identify the “linearization” of $E_*^\wedge \mathcal{B}(D)$ with the bar complex $\mathcal{B}(\Delta)$. We also need this fact in order to show that $\mathcal{B}(\Delta)$ is torsion free; this is used to carry out the reduction (8.8) from the calculation of the partition complex $\tilde{Q}(\overline{P}_m)$ to the calculation of the uniform complex $\tilde{Q}(\overline{U}_m)$.

1.2. Historical remarks. The classical example of an algebra of power operations which is Koszul is the May-Dyer-Lashof algebra of power operations in the homology of an differential graded E_∞ -algebra over \mathbb{F}_p . That this algebra is Koszul appears to be well-known, though I don’t know an explicit reference; it is implicitly proved in [AM99, §3.1]. The proof for the prime 2 is an application of the PBW basis theorem of Priddy; an adjustment needs to be made to give a proof at odd primes. Kuhn has an elegant unpublished proof that this algebra is Koszul (at least at the prime 2) which bypasses the need to find an admissible basis.

Power operations were first constructed for Morava E -theory by Ando [And95], using power operations for MU and suitable choices of orientations. Soon after, Hopkins and

Miller were able to show that Morava E -theories are E_∞ -ring spectra, although it took a long time for the technical details to be worked out. Further work by Ando, Hopkins, and Strickland allows for a description of the ring Γ of power operations, in terms of the relevant part of the E -cohomology of symmetric groups. The key result here is Strickland's identification of a quotient of $E^*B\Sigma_{p^r}$ as the ring classifying subgroups of a formal group [Str97], [Str98]. Some exposition of these results is given in [Rez09].

The ring of power operations for height 1 Morava E -theories amounts to the case of p -adic K -theory; this case is understood by work of McClure [McC83]. The first (partial) calculation of a power operations algebra for height 2 was carried out by Kashiwabara [Kas95]. What he really did is find a basis for the Morava $K(2)$ -homology of symmetric groups. In our language he did this by understanding $\Gamma/(p, v_1)\Gamma$. The ideal $(p, v_1)\Gamma \subset \Gamma$ is not a two-sided ideal, so $\Gamma/(p, v_1)$ is not actually a ring. Thus Kashiwabara (aware of this) only computed a product up an indeterminacy. His calculations nonetheless make clear that at height 2, the ring Γ is a quadratic algebra, and that Γ has an “admissible basis” in terms of certain monomials in the generators, and that the algebra should satisfy the PBW condition of Priddy; thus Γ is Koszul at height 2.

Ando, Hopkins, and Strickland conjectured that there is a small resolution (of length n) for modules over the ring of power operations of a height n Morava E -theory, for any n ; that is, that these rings are Koszul. Moreover, they explicitly describe a model for this resolution; the description involves the “building complex” of the finite subgroup schemes of a formal group. A brief discussion of these ideas are given in [Str97, §14]. We do not address their “building complex” construction in this paper; however, in [Rez12] we have described a version of the building complex for height $n = 2$, using elliptic curves.

I announced the theorem of this paper in a talk in Mainz in 2005. I later realized that the proof I believed I had at that time was not complete; in particular, my argument to prove (8.8) was incorrect. I found a correct proof in 2008. This was one of the factors delaying the appearance of this paper, another being the need to map out the foundations of the theory of power operations for Morava E -theory, some of which was done in [Rez09].

1.3. Acknowledgments. The ideas in the proof given here owe a great deal to the work of Arone and Mahowald on the Goodwillie tower of the identity functor [AM99]. I am of course indebted to the work of Ando, Hopkins, and Strickland on power operations for Morava E -theory, which is the foundation of the present work.

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2. MONADS AND BAR CONSTRUCTIONS

2.1. Exponential monads. Let \mathcal{C} be a symmetric monoidal category with monoidal product \otimes and unit $\mathbb{1}$, and suppose also that \mathcal{C} admits finite coproducts (denoted “ \oplus ”, with initial object 0), and that \otimes distributes over coproducts. For convenience, we also assume that inclusions of direct summands are always monomorphisms in \mathcal{C} . By an **exponential monad**, we mean a monad equipped with natural isomorphisms

$$v: \mathbb{1} \rightarrow T(0), \quad \zeta: T(X) \otimes T(Y) \rightarrow T(X \oplus Y),$$

where the map ζ is a natural transformation of functors $\mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$, with the property that (v, ζ) make $T: \mathcal{C}^\oplus \rightarrow \mathcal{C}^\otimes$ into a strong symmetric monoidal functor. Furthermore, we require that every T -algebra $(A, \phi: TA \rightarrow A)$ is naturally a commutative monoid object in the symmetric monoidal category \mathcal{C} , with unit $\mathbb{1} \xrightarrow{v} T(0) \xrightarrow{T(0)} T(A) \xrightarrow{\psi} A$ and product $A \otimes A \xrightarrow{\eta \otimes \eta} TA \otimes TA \xrightarrow{\zeta} T(A \oplus A) \xrightarrow{T(\nabla)} TA \xrightarrow{\psi} A$.

The canonical example of an exponential monad is the free commutative algebra monad on the category of abelian groups. The examples we need to work with will be free E_∞ -algebra monads on some homotopy category of spaces or spectra, or monads derived from such.

2.2. Graded exponential monads. By a **graded exponential monad**, we mean an exponential monoidal monad T , together with functors $T\langle m \rangle: \mathcal{C} \rightarrow \mathcal{C}$ and natural monomorphisms $\gamma_m: T\langle m \rangle(X) \rightarrow T(X)$, which fit together to give a direct sum decomposition

$$(\gamma_m): \bigoplus_{m \geq 0} T\langle m \rangle(X) \xrightarrow{\sim} T(X)$$

and such that there exist (necessarily unique, because the γ_k are monomorphisms) dotted arrows in

$$\begin{array}{ccc} \mathbb{1} \xrightarrow{\dots \eta_0} T\langle 0 \rangle(0) & T\langle p \rangle(X) \otimes T\langle q \rangle(Y) \xrightarrow{\dots \zeta_{p,q}} T\langle p+q \rangle(X \oplus Y) \\ \searrow v \downarrow \gamma_0 & \downarrow \gamma_p \otimes \gamma_q \downarrow \gamma_{p+q} \\ T(0) & T(X) \otimes T(Y) \xrightarrow{\zeta} T(X \oplus Y) \end{array}$$

$$\begin{array}{ccc} X \xrightarrow{\dots \eta_1} T\langle 1 \rangle(X) & T\langle p \rangle T\langle q \rangle(X) \xrightarrow{\dots \mu_{p,q}} T\langle pq \rangle(X) \\ \searrow \eta \downarrow \gamma_1 & \downarrow \gamma_p \gamma_q \downarrow \gamma_{pq} \\ T(X) & TT(X) \xrightarrow{\mu} T(X) \end{array}$$

such that $T\langle 0 \rangle(0) \rightarrow T\langle 0 \rangle(X)$ and $\eta_1: X \rightarrow T\langle 1 \rangle(X)$ are isomorphisms for all objects X . These conditions imply that

$$\mathbb{1} \rightarrow T\langle 0 \rangle(X), \quad (\zeta_{p,q}): \bigoplus_{p+q=m} T\langle p \rangle(X) \otimes T\langle q \rangle(Y) \rightarrow T\langle p+q \rangle(X \oplus Y)$$

are isomorphisms. Furthermore, each composite $T^{\circ q}$ admits a direct sum decomposition $T^{\circ q} \approx \bigoplus_{m \geq 0} T^{\circ q}\langle m \rangle$, determined inductively by

$$T^{\circ q}\langle m \rangle(X) \approx \bigoplus_{m = \sum_j j m_j} \left[\bigotimes_{j \geq 0} T\langle m_j \rangle(T^{\circ(q-1)}\langle j \rangle(X)) \right],$$

and this decomposition is compatible with the structure maps of the monad. That is, each map $T^{\circ i} \circ \mu \circ T^{\circ j}: T^{\circ(i+j+1)} \rightarrow T^{\circ(i+j)}$ restricts to a coproduct of maps $T^{\circ(i+j+1)}\langle m \rangle \rightarrow T^{\circ(i+j)}\langle m \rangle$, and similarly for $T^{\circ i} \circ \eta \circ T^{\circ j}$.

We refer to $T^{\circ q}\langle m \rangle$ as the **weight** m part of $T^{\circ q}$. Note that if $m = \prod_{i=1}^q m_i$, then there is an evident map

$$T\langle m_1 \rangle \circ \cdots \circ T\langle m_q \rangle \rightarrow T^{\circ q}\langle m \rangle,$$

which is an inclusion of a direct summand. We say that summands of this form have **pure weight** m .

2.3. The positive part of a graded exponential monad. Given a graded exponential monad T on \mathcal{C} , we write \tilde{T} for the subfunctor $\tilde{T}(X) = \bigoplus_{m \geq 1} T\langle m \rangle(X)$ consisting of the part in positive weight. It is clear that \tilde{T} is a monad in its own right, so that the inclusion map $\tilde{T} \rightarrow T$ is a map of monads. We can similarly speak of the weight m part $\tilde{T}^{\circ q}\langle m \rangle$ of $\tilde{T}^{\circ q}$, which will be a subobject of $T^{\circ q}\langle m \rangle$. (Note that $\tilde{T}^{\circ q}\langle m \rangle$ will not generally be equal to $T^{\circ q}\langle m \rangle$ if $q \geq 2$, as the latter contains a large number of contributions from the weight 0 part of T .)

2.4. Bar complexes. Given a monad $(T, \eta: I \rightarrow T, \mu: T \circ T \rightarrow T)$ on a category \mathcal{C} , we will write $\mathcal{B}(T) = \mathcal{B}(T, T, T)$ for the two-sided bar construction for T ; this is an augmented simplicial object in endofunctors of \mathcal{C} , with $\mathcal{B}_q(T) = T^{\circ(q+2)}$. More generally, given $M, N: \mathcal{C} \rightarrow \mathcal{C}$ which are left and right modules for T respectively, there is a bar construction $\mathcal{B}(M, T, N)$ with $\mathcal{B}_q(M, T, N) \approx M \circ T^{\circ q} \circ N$.

Now suppose T is a graded exponential monad. For each $q \geq 0$ and $m \geq 0$ we define

$$\mathcal{B}_q(T)\langle m \rangle \stackrel{\text{def}}{=} T^{\circ(q+2)}\langle m \rangle,$$

using the direct sum decomposition $T^{\circ(q+2)} \approx \bigoplus_{m \geq 0} T^{\circ(q+2)}\langle m \rangle$ described above. Thus, the simplicial endofunctor $\mathcal{B}(T) \approx \bigoplus_{m \geq 0} \mathcal{B}(T)\langle m \rangle$ admits a weight decomposition.

We may similarly consider the bar construction of the positive part \tilde{T} , and we similarly obtain a weight decomposition $\mathcal{B}(\tilde{T}) \approx \bigoplus_{m \geq 1} \mathcal{B}(\tilde{T})\langle m \rangle$.

2.5. Example. Let O be an E_∞ -operad in spaces, and let C denote the monad on spaces defined by $C(X) \stackrel{\text{def}}{=} \coprod_{m \geq 0} (O(m) \times X^m)_{h\Sigma_m}$. The functor C descends to a monad on the homotopy category $h\text{Spaces}$ of spaces, which we also denote by C . This is a graded exponential monad in our sense; the graded pieces are $C\langle m \rangle(X) \approx (O(m) \times X^m)_{h\Sigma_m}$.

The corresponding bar complex admits a grading $\mathcal{B}(C) \approx \coprod_{m \geq 0} \mathcal{B}(C)\langle m \rangle$; applied to a space, we obtain a decomposition $\mathcal{B}(C)(X) \approx \coprod_{m \geq 0} \mathcal{B}(C)\langle m \rangle(X)$ of simplicial spaces. As we will note below (7.11), for the positive part of C there is a natural weak equivalence

$$\mathcal{B}(\tilde{C})\langle m \rangle(X) \approx (P_m \times X^m)_{h\Sigma_m}$$

of simplicial spaces, where P_m is the partition complex (6.2) on the set of m elements.

2.6. Example. The monad D on the homotopy category of spectra, defined by $D(Y) \stackrel{\text{def}}{=} \bigvee_{m \geq 0} (O(m)_+ \wedge Y^{\wedge m})_{h\Sigma_m}$, is similarly an exponential monad.

2.7. Example. Let \mathbb{P} denote the free commutative E -algebra monad, defined on the category Mod_E of E -module spectra. This functor descends to a functor on the homotopy category $h\text{Mod}_E$, which we also denote by \mathbb{P} . As such, it is a graded exponential functor, with $\mathbb{P} \approx \bigvee_{m \geq 0} \mathbb{P}\langle m \rangle$, where $\mathbb{P}\langle m \rangle(M) \approx (M^{\wedge_{E^m}})_{h\Sigma_m}$. We will typically write \mathbb{P}_m for $\mathbb{P}\langle m \rangle$.

2.8. Associations. Let T and T' be graded exponential monads on suitable categories \mathcal{C} and \mathcal{C}' . An **association** from T to T' is a functor $G: \mathcal{C} \rightarrow \mathcal{C}'$ which is equipped with the structure of a weak monoidal functor in two different ways, namely as functors $C_{\oplus} \rightarrow C'_{\oplus}$ and $C_{\otimes} \rightarrow C'_{\otimes}$, together with a natural map $TG \rightarrow GT'$ which is compatible with all the structure.

2.9. Example. The functor $\Sigma_+^{\infty}: h\text{Spaces} \rightarrow h\text{Spectra}$ defines an association between C and D . Likewise, the functor $\Sigma: h\text{Spectra} \rightarrow h\text{Spectra}$ defines an association between D and itself.

2.10. Example. There is an association between the monads C and \mathbb{P} described above, given by the functor $E \wedge \Sigma_+^{\infty}: h\text{Spaces} \rightarrow h\text{Mod}_E$. In particular: (i) $E \wedge \Sigma_+^{\infty}$ takes coproducts to coproducts; (ii) $E \wedge \Sigma_+^{\infty}$ takes products to smash products; (iii) there is a natural map (in fact, a weak equivalence)

$$\mathbb{P}(E \wedge \Sigma_+^{\infty} X) \rightarrow E \wedge \Sigma_+^{\infty} C(X),$$

which is compatible with the exponential structures on the monads, and which is compatible with the gradings, in the sense that it restricts to maps $E \wedge \Sigma_+^{\infty} C\langle m \rangle(X) \rightarrow \mathbb{P}\langle m \rangle(X)$.

2.11. Example. One more example of exponential monad is given in the next section, where we describe a monad \mathbb{T} on the category Mod_{E_*} of E_* -modules. There is an association between \mathbb{T} and the monad \mathbb{P} above, given by $\pi_* L_{K(n)}: h\text{Mod}_E \rightarrow \text{Mod}_{E_*}$, so that there is a natural map

$$\mathbb{T}(\pi_* L_{K(n)} M) \rightarrow \pi_* L_{K(n)} \mathbb{P}(M).$$

3. RINGS OF POWER OPERATIONS

The homotopy groups of a $K(n)$ -local commutative E -algebra spectrum are naturally algebras over a certain monad \mathbb{T} , which captures algebraically the E -homology of symmetric groups. In this section, we recall from [Rez09] properties of the monad \mathbb{T} . From this monad, we will extract two kinds of graded rings of “power operations”; the ring Γ^q , which is a ring of *additive* operations on π_{-q} of a $K(n)$ -local commutative E -algebra, and the ring Δ^q , which is a ring of operations on the degree $-q$ part of the cotangent space of an augmented algebra. The main result of this section is that all of the rings in question are isomorphic. It is the ring $\Delta \stackrel{\text{def}}{=} \Delta^0$ which we will explicitly show is Koszul in subsequent sections.

In this section, we fix a Morava E -theory spectrum, with height n .

3.1. Completed E -homology. Given a spectrum Y , we define the **completed E -homology** of Y by

$$E_*^{\wedge} Y \stackrel{\text{def}}{=} \pi_* L_{K(n)}(E \wedge Y).$$

If X is a space, we write $E_*^{\wedge} X$ for $E_*^{\wedge} \Sigma_+^{\infty} X$. As is well-known, the functor E_*^{\wedge} satisfies the Eilenberg-Steenrod axioms but not Milnor’s wedge axiom.

3.2. Algebraic structure on the homotopy groups of E -algebras. As we noted above (2.7), the free commutative E -algebra functor $\mathbb{P}: h\text{Mod}_E \rightarrow h\text{Mod}_E$ carries the structure of a graded exponential monad with respect to derived smash product of E -modules; in particular, it admits a decomposition $\mathbb{P} \approx \bigvee_{m \geq 0} \mathbb{P}_m$. We need one more piece of structure;

namely, for all $m \geq 1$, there is a natural transformation $e: \Sigma \mathbb{P}_m \rightarrow \mathbb{P}_m \Sigma$, defined because \mathbb{P}_m is compatible with the enrichment of Mod_E over pointed spaces.

In [Rez09, §4] we described an **algebraic approximation functor** $\mathbb{T}: \text{Mod}_{E_*} \rightarrow \text{Mod}_{E_*}$. The purpose of this functor is to mirror algebraically as closely as possible the functor \mathbb{P} , by means of an **approximation map**

$$\alpha: \mathbb{T}(\pi_* L_{K(n)} M) \rightarrow \pi_* L_{K(n)}(\mathbb{P} M),$$

natural in $M \in h\text{Mod}_E$. We summarize here the salient properties of the algebraic approximation.

- (1) The functor \mathbb{T} is equipped with the structure of an exponential monad with respect to tensor product of E_* -modules [Rez09, Prop. 4.10, §4.13], and the approximation map [Rez09, §4.6] is an association between \mathbb{T} and \mathbb{P} in the sense of (2.8). (This is clear from the construction of γ_k in [Rez09, §4.13].)
- (2) The functor \mathbb{T} is a *graded* exponential monad, with decomposition $\mathbb{T} = \bigoplus_{m \geq 0} \mathbb{T}\langle m \rangle$ [Rez09, §4.4, §4.13]; typically we write \mathbb{T}_m for $\mathbb{T}\langle m \rangle$. Furthermore, the approximation map descends to the grading [Rez09, §4.6], in the sense that α factors through a coproduct of maps of the form

$$\alpha_m: \mathbb{T}_m(\pi_* L_{K(n)} M) \rightarrow \pi_* L_{K(n)}(\mathbb{P}_m M).$$

- (3) If M is an E -module such that $\pi_* M$ is finite and free as an E_* -module, then α_m evaluated at M is an isomorphism [Rez09, Prop. 4.8].
- (4) If N is a finite and free E_* -module, then so is $\mathbb{T}_m N$ for all $m \geq 0$ (by (3) and [Rez09, Prop. 3.16]). Thus for all $q \geq 0$, the approximation maps induce isomorphisms

$$\mathbb{T}^{\circ q}\langle m \rangle(\pi_* L_{K(n)} M) \xrightarrow{\sim} \pi_* L_{K(n)}(\mathbb{P}^{\circ q}\langle m \rangle M)$$

when $\pi_* M$ is finite and free.

- (5) Each functor $\mathbb{T}_m: \text{Mod}_{E_*} \rightarrow \text{Mod}_{E_*}$ is the left Kan extension of its own restriction to the full subcategory of finite free E_* -modules in Mod_{E_*} . The functor $\mathbb{T} = \bigoplus \mathbb{T}_m$ is also a Kan extension of its restriction. (This is the construction of these functors given in [Rez09, §4.4].)
- (6) The functor \mathbb{T} and each functor \mathbb{T}_m commutes with filtered colimits and reflexive coequalizers [Rez09, Prop. 4.12].
- (7) If M is an E_* -module concentrated in even degree, then so is $\mathbb{T}M$. (This follows using the Kan extension property (5), the isomorphism (3), and the fact that $E_*^\wedge B\Sigma_m$ are concentrated in even degree.)
- (8) Let $\text{Alg}_{\mathbb{T}}^*$ denote the category of algebras for the monad \mathbb{T} on Mod_{E_*} . This category is complete and cocomplete; limits are computed in the underlying category Mod_{E_*} , and colimits are computed in the underlying category Alg_{E_*} of commutative E_* -algebras. In particular, coproducts in $\text{Alg}_{\mathbb{T}}^*$ are tensor products of E_* -modules [Rez09, Cor. 4.19].
- (9) Using the approximation map, we see that every algebra A for the monad \mathbb{P} on $h\text{Mod}_E$ (i.e., for every H_∞ - E -algebra), the homotopy groups $\pi_* L_{K(n)} A$ naturally carry the structure of a \mathbb{T} -algebra. That is, we obtain a functor

$$\pi_* L_{K(n)}: \text{Alg}_{\mathbb{P}} \rightarrow \text{Alg}_{\mathbb{T}}^*$$

lifting $\pi_* L_{K(n)}: \text{Mod}_E \rightarrow \text{Mod}_{E_*}$.

- (10) Let $\Sigma: \text{Mod}_{E_*} \rightarrow \text{Mod}_{E_*}$ denote the functor $\Sigma M \stackrel{\text{def}}{=} E_* S^1 \otimes_{E_*} M$. For $m \geq 1$, there is an algebraic **suspension map**

$$E: \Sigma \mathbb{T}_m \rightarrow \mathbb{T}_m \Sigma,$$

which with respect to the approximation map is compatible with the topological suspension map $e: \Sigma \mathbb{P}_m \rightarrow \mathbb{P}_m \Sigma$. Furthermore, the algebraic suspension map is compatible with the monad structure on \mathbb{T} , in the sense that both ways of building a map $\Sigma(\mathbb{T}\mathbb{T})\langle m \rangle \rightarrow (T\langle m \rangle)\Sigma$ coincide. (This suspension map is denoted E_1 in [Rez09, §4.25], where the notation $\omega^{-q/2}$ is used for $E_* S^q$. Compatibility between algebraic and topological suspension maps is by construction. The algebraic suspension map is defined for finite free E_* -modules by means of the approximation isomorphism and the topological suspension map; the general algebraic suspension map is then defined using (5) above. The compatibility with the monad structure is [Rez09, Prop. 4.27].)

We note that the full approximation map $\alpha: \mathbb{T}(\pi_* L_{K(n)} M) \rightarrow \pi_* L_{K(n)}(\mathbb{P}M)$ is almost never an isomorphism, due to the fact that $L_{K(n)}$ does not commute with arbitrary coproducts.

We need one more fact about the algebraic suspension.

3.3. Proposition. *For $i, j \geq 1$, the composition*

$$\Sigma(\mathbb{T}_i M \otimes \mathbb{T}_j N) \xrightarrow{\Sigma \zeta} \Sigma \mathbb{T}_{i+j}(M \oplus N) \xrightarrow{E} \mathbb{T}_{i+j} \Sigma(M \oplus N)$$

is equal to 0.

Proof. Consider functors $\text{Mod}_E \rightarrow \text{Mod}_E$ defined by $(X, Y) \mapsto \mathbb{P}_i X \wedge_E \mathbb{P}_j Y$ and $(X, Y) \mapsto \mathbb{P}_{i+j}(X \vee Y)$. These are each enriched over pointed spaces, and thus we obtain a homotopy commutative diagram

$$\begin{array}{ccc} \Sigma(\mathbb{P}_i X \wedge \mathbb{P}_j Y) & \longrightarrow & \mathbb{P}_i \Sigma X \wedge \mathbb{P}_j \Sigma Y \\ \Sigma \zeta \downarrow & & \downarrow \\ \Sigma \mathbb{P}_{i+j}(X \vee Y) & \xrightarrow{e} & \mathbb{P}_{i+j}(\Sigma X \vee \Sigma Y) \end{array}$$

where the vertical maps come from the exponential structure. We see that the bottom horizontal map is the suspension map e , while the top horizontal map factors $\Sigma(\mathbb{P}_i X \wedge \mathbb{P}_j Y) \rightarrow \Sigma \mathbb{P}_i X \wedge \Sigma \mathbb{P}_j Y \rightarrow \mathbb{P}_i \Sigma X \wedge \mathbb{P}_j \Sigma Y$. The first of these two maps is null, as it uses the diagonal embedding $S^1 \rightarrow S^1 \wedge S^1$. Thus the composite $e \circ \zeta$ is null. The corresponding vanishing result for the algebraic suspension map is immediate for finite free modules, and follows in general since the algebraic approximation functors are left Kan extended from finite free modules. \square

3.4. Augmented rings. Let $\text{Alg}_{\mathbb{T}}^*/E_*$ denote the category of \mathbb{T} -objects augmented over E_* ; an object of $\text{Alg}_{\mathbb{T}}^*/E_*$ is a morphism $A \rightarrow E_*$.

We will write $\tilde{\mathbb{T}}$ for the subfunctor of \mathbb{T} defined by

$$\tilde{\mathbb{T}}(M) \stackrel{\text{def}}{=} \bigoplus_{m \geq 1} \mathbb{T}_m(M).$$

This is precisely the kernel of the natural augmentation map $\mathbb{T}(M) \rightarrow \mathbb{T}(0) = E_*$.

As noted above (§2.3), the functor $\tilde{\mathbb{T}}$ itself inherits the structure of a monad on Mod_{E_*} . The category of algebras over $\tilde{\mathbb{T}}$ is equivalent to the category of $\text{Alg}_{\tilde{\mathbb{T}}}/E_*$ of augmented \mathbb{T} -algebras. This is a standard observation, so we won't spell out the details, except to note that if $A \rightarrow E_*$ is an augmented \mathbb{T} -algebra with structure map $\psi: \mathbb{T}A \rightarrow A$ and augmentation ideal \tilde{A} , then the corresponding $\tilde{\mathbb{T}}$ -algebra structure $\tilde{\psi}: \tilde{\mathbb{T}}\tilde{A} \rightarrow \tilde{A}$ is simply the restriction of ψ to $\tilde{\mathbb{T}}\tilde{A} \subset \mathbb{T}A$.

3.5. Abelian group objects. Recall that the notion of **abelian group object** can be defined in any category with finite products.

3.6. Proposition. *An object $A \rightarrow E_*$ of $\text{Alg}_{\tilde{\mathbb{T}}}/E_*$ with augmentation ideal \tilde{A} admits the structure of an abelian group object if and only if $\tilde{A}^2 = 0$, in which case the abelian group structure is unique.*

Proof. An abelian group structure is a map $f: A \times_{E_*} A \rightarrow A$ of \mathbb{T} -algebras, which satisfies the axioms for an abelian group; the unit of the abelian group is necessarily given by the unique \mathbb{T} -algebra map $E_* \rightarrow A$. Since $A \approx E_* \oplus \tilde{A}$ as E_* -modules, we see that the evident \mathbb{T} -algebra map $A \otimes_{E_*} A \rightarrow A \times_{E_*} A$ is surjective, with kernel $\tilde{A} \otimes_{E_*} \tilde{A}$. Thus, a map f satisfying the unit axiom for an abelian group exists if and only if the multiplication map $A \otimes_{E_*} A \rightarrow A$ sends $\tilde{A} \otimes_{E_*} \tilde{A}$ to 0. That is, a unital f exists if and only if $\tilde{A}^2 = 0$. It is straightforward to show that if such a unital f exists, it is given by $f(c, x, y) = c + x + y$ (written in terms of the E_* -module decompositions $A \approx E_* \oplus \tilde{A}$ and $A \times_{E_*} A \approx E_* \oplus \tilde{A} \oplus \tilde{A}$), and therefore is the unique abelian group structure on A . \square

Thus, the category $(\text{Alg}_{\tilde{\mathbb{T}}}/E_*)_{\text{ab}}$ of abelian group objects is identified with the full subcategory $\text{Alg}_{\tilde{\mathbb{T}}}/E_*$ of augmented \mathbb{T} -algebras with $\tilde{A}^2 = 0$, and the left adjoint to this inclusion is given by $A \mapsto A/\tilde{A}^2$, which can be regarded as providing the **cotangent space** to A at the augmentation $A \rightarrow E_*$.

3.7. Suspension and loop. The homotopy category $h(\text{Alg}_E/E)$ of augmented commutative E -algebra spectra admits a **loop** functor $\Omega: h(\text{Alg}_E/E) \rightarrow h(\text{Alg}_E/E)$. If A is an augmented commutative E -algebra, then ΩA is the homotopy pullback

$$\begin{array}{ccc} \Omega A & \longrightarrow & E \\ \downarrow & & \downarrow \\ E & \longrightarrow & A \end{array}$$

in $h(\text{Alg}_E/E)$. The underlying E -module spectrum of ΩA has the form $E \vee \Sigma^{-1}\tilde{A}$, where \tilde{A} is the homotopy fiber of the augmentation $A \rightarrow E$.

There is a corresponding loop functor $\Omega: \text{Alg}_{\tilde{\mathbb{T}}}/E_* \rightarrow \text{Alg}_{\tilde{\mathbb{T}}}/E_*$, with the property that as an E_* -module $\Omega A \approx E_* \oplus (E_* S^{-1} \otimes_{E_*} \tilde{A})$, where \tilde{A} is the augmentation ideal. Furthermore, the augmentation ideal of ΩA will be square-zero, and so Ω factors through a functor $\text{Alg}_{\tilde{\mathbb{T}}}/E_* \rightarrow (\text{Alg}_{\tilde{\mathbb{T}}}/E_*)_{\text{ab}}$.

To define the loop functor, recall the suspension map $E: \Sigma \tilde{\mathbb{T}} \rightarrow \tilde{\mathbb{T}} \Sigma$ of §3.2(10). Since $\Sigma: \text{Mod}_{E_*} \rightarrow \text{Mod}_{E_*}$ is a self equivalence, it has an inverse functor Σ^{-1} , and we may therefore use it to define the **desuspension map** $E': \tilde{\mathbb{T}} \Sigma^{-1} \rightarrow \Sigma^{-1} \tilde{\mathbb{T}}$.

Let $A \rightarrow E_*$ in $\text{Alg}_{\mathbb{T}}^*/E_*$ with augmentation ideal \tilde{A} . As noted in §3.4, it is equivalent to consider \tilde{A} as an algebra over $\tilde{\mathbb{T}}$, with structure map $\tilde{\psi}: \tilde{\mathbb{T}} \tilde{A} \rightarrow \tilde{A}$. We thus *define* ΩA to be the augmented \mathbb{T} -algebra with underlying augmentation ideal $\Sigma^{-1} \tilde{A}$, and with structure map $\tilde{\psi}_\Omega$ defined as the composite

$$\tilde{\mathbb{T}} \Sigma^{-1} \tilde{A} \xrightarrow{E'} \Sigma^{-1} \tilde{\mathbb{T}} \tilde{A} \xrightarrow{\Sigma^{-1} \tilde{\psi}} \tilde{A}.$$

To show that $\tilde{\psi}_\Omega$ defines a $\tilde{\mathbb{T}}$ -algebra structure is a straightforward calculation, which relies on the compatibility of suspension with the monad structure of \mathbb{T} described in 3.2(10). That the augmentation ideal of ΩA is square-zero amounts to (3.3).

3.8. The rings Γ^q . The ring Γ^q is defined to be a ring which naturally acts on the degree $(-q)$ -part of the underlying E_0 -module of an \mathbb{T} -algebra, and hence acts naturally on π_{-q} of a $K(n)$ -local commutative E -algebra.

Let $U^q: \text{Alg}_{\mathbb{T}}^* \rightarrow \text{Ab}$ denote the functor which sends a \mathbb{T} -algebra A to its $(-q)$ -th grading A_q , viewed as an abelian group. We *define* $\Gamma^q \stackrel{\text{def}}{=} \text{End}(U^q)$, the endomorphism ring of the functor U^q ; thus, an element $f \in \Gamma^q$ gives a natural abelian group homomorphism $A_{-q} \rightarrow A_{-q}$ for all A in $\text{Alg}_{\mathbb{T}}^*$.

The underlying set of $U^q(A)$ is naturally isomorphic to $\text{Hom}_{\text{Alg}_{\mathbb{T}}^*}(\mathbb{T}(E_* S^{-q}), A)$, where $\mathbb{T}(M)$ represents the free \mathbb{T} -algebra on an E_* -module M . Therefore, we see that endomorphisms of the composite functor $\text{Alg}_{\mathbb{T}}^* \xrightarrow{U^q} \text{Ab} \rightarrow \text{Set}$ are exactly the \mathbb{T} -algebra endomorphisms of $\mathbb{T}(E_* S^{-q})$. Hence, the monoid of set-endomorphisms of U^q is

$$\text{Hom}_{\text{Alg}_{\mathbb{T}}^*}(\mathbb{T}(E_* S^{-q}), \mathbb{T}(E_* S^{-q})) \approx \text{Hom}_{\text{Mod}_{E_0}}(E_* S^{-q}, \mathbb{T}(E_* S^{-q})) \approx \mathbb{T}(E_* S^{-q})_{-q},$$

and so Γ^q may be identified with the degree $-q$ *primitives* of $\mathbb{T}(E_* S^{-q})$. We thus recover the definition of Γ^q given in [Rez09, §7], where the notation $\omega^{q/2}$ is used for the E_* -module $E_* S^{-q}$.

Using the isomorphisms

$$\mathbb{T}_m(E_* S^{-q}) \approx \pi_* L_{K(n)} \mathbb{P}_m(E \wedge S^{-q}) \approx E_*^\wedge B \Sigma_m^{-q \rho_m}$$

and

$$\mathbb{T}_m(E_* S^{-q} \oplus E_* S^{-q}) \approx \pi_* L_{K(n)} \mathbb{P}_m(E \wedge (S^{-q} \vee S^{-q})) \approx \bigoplus_{0 \leq j \leq m} E_*^\wedge B(\Sigma_j \times \Sigma_{m-j})^{-q \rho_m},$$

we find that $\Gamma^q \approx \bigoplus_{k \geq 0} \Gamma^q[k]$, where

$$\Gamma^q[k] \approx \text{Ker} \left[E_{-q}^\wedge B \Sigma_{p^k}^{-q \rho_{p^k}} \rightarrow \bigoplus_{0 < j < p^k} E_{-q}^\wedge B(\Sigma_j \times \Sigma_{p^k-j})^{-q \rho_{p^k}} \right],$$

the map being induced by the transfer map associated to the inclusion $\Sigma_j \times \Sigma_{p^k-j} \subset \Sigma_{p^k}$.

We note the following important fact, which is proved as [Rez09, Prop. 7.3], though it ultimately derives from [Str98]. The statement about rank can be read off from calculations

of $E_*^\wedge B\Sigma_m^{-q\rho_m} \otimes \mathbb{Q}$, or alternately from calculations of the Morava K -theory of the free commutative S -algebras DS^0 and DS^1 , as in the proof of [Str98, Prop. 5.6].

3.9. Proposition. *Each $\Gamma^q[k]$ is a finitely generated free E_* -module, and the rank of $\Gamma^q[k]$ does not depend on the dimension q .*

Here is a variant of the above description of Γ^q , which we will need below. Let $I^q: \text{Alg}_{\mathbb{T}}^*/E_* \rightarrow \text{Ab}$ denote the functor which sends an augmented \mathbb{T} -algebra to the $(-q)$ -degree part of its augmentation ideal. There is an evident ring homomorphism $\text{End}(U^q) \rightarrow \text{End}(I^q)$, and it is straightforward to show that this is an isomorphism. That is, Γ^q is also the endomorphism ring of I^q .

3.10. The rings Δ^q . The ring Δ^q is defined to be a ring which acts naturally on the degree $(-q)$ -part of the *cotangent space* to an augmentation $A \rightarrow E_*$.

Let $V^q: \text{Alg}_{\mathbb{T}}^*/E_* \rightarrow \text{Ab}$ denote the functor which sends an augmented \mathbb{T} -algebra $A \rightarrow E_*$ to the $(-q)$ -th grading of its abelianization \tilde{A}/\tilde{A}^2 , where \tilde{A} is the augmentation ideal. We define $\Delta^q \stackrel{\text{def}}{=} \text{End}(V^q)$, the endomorphism ring of the functor V^q ; thus an element $f \in \Delta^q$ gives a natural abelian group homomorphism $(\tilde{A}/\tilde{A}^2)_{-q} \rightarrow (\tilde{A}/\tilde{A}^2)_{-q}$ for all A in $\text{Alg}_{\mathbb{T}}^*/E_*$.

To each endomorphism $\phi: V^q \rightarrow V^q$ we associate an element $\phi(x)$ in the E_* -module $(\tilde{\mathbb{T}}(E_*S^{-q})/(\tilde{\mathbb{T}}(E_*S^{-q}))^2)_{-q}$, defined as the image of the canonical generator of $E_{-q}S^{-q}$ under the map

$$x \in E_{-q}S^{-q} \rightarrow \tilde{\mathbb{T}}(E_*S^{-q})_{-q} \rightarrow V^q(\mathbb{T}(E_*S^{-q})) \xrightarrow{\phi} V^q(\mathbb{T}(E_*S^{-q}));$$

3.11. Proposition. *The map $\Delta^q \rightarrow V^q(\mathbb{T}(E_*S^{-q})) \approx (\tilde{\mathbb{T}}(E_*S^{-q})/(\tilde{\mathbb{T}}(E_*S^{-q}))^2)_{-q}$ sending ϕ to $\phi(x)$ is an isomorphism.*

Proof. It is straightforward to check, using naturality and the bijection $\tilde{A}_{-q} = \text{Hom}_{\text{Alg}_{\mathbb{T}}^*/E_*}(\mathbb{T}(E_*S^{-q}), A)$, that an endomorphism ϕ is uniquely determined by the element $\phi(x)$. Thus it remains to show that the map of the proposition is surjective.

Next, suppose that ϕ is an endomorphism of the composite functor $\text{Alg}_{\mathbb{T}}^*/E_* \xrightarrow{V^q} \text{Ab} \rightarrow \text{Set}$. The abelian group structure $V^q(A) \times V^q(A) \rightarrow V^q(A)$ is naturally isomorphic to the map obtained by applying V^q to the fold map $A \otimes_{E_*} A \rightarrow A$ of augmented \mathbb{T} -algebras. Thus, by naturality, ϕ must commute with this map. That is, every set endomorphism of V^q is automatically an abelian group endomorphism.

Given $y \in V^q(\mathbb{T}(E_*S^{-q}))$, choose any lift $y \in \tilde{\mathbb{T}}(E_*S^{-q})_{-q}$ and consider the corresponding endomorphism ψ of the composite functor $\text{Alg}_{\mathbb{T}}^*/E_* \xrightarrow{I^q} \text{Ab} \rightarrow \text{Set}$, which sends an augmented algebra to the underlying set of its augmentation ideal. Because $A \mapsto A/\tilde{A}^2$ is a functor from $\text{Alg}_{\mathbb{T}}^*/E_*$ to itself, we can apply ψ to $I^q(A/\tilde{A}^2) \approx V^q(A)$ to obtain a natural abelian group endomorphism of V^q , and a straightforward calculation shows that the evaluation of this endomorphism on the canonical generator is exactly y , as desired. \square

Thus, the E_0 -module Δ^q is isomorphic to the indecomposable quotient of the augmented E_* -algebra $\mathbb{T}(E_*S^{-q})_{-q} \approx \bigoplus_{m \geq 0} E_{-q}^\wedge B\Sigma_m^{-q\rho_m}$. The rings Δ^q admit a grading

$\Delta^q \approx \bigoplus_{k \geq 0} \Delta^q[k]$, where

$$\Delta^q[k] \approx \text{Cok} \left[\bigoplus_{0 < j < p^k} E_{-q}^\wedge B(\Sigma_j \times \Sigma_{p^k-j})^{-q\rho_{p^k}} \rightarrow E_{-q}^\wedge B\Sigma_{p^k}^{-q\rho_{p^k}} \right],$$

the map being induced by the inclusion $\Sigma_j \times \Sigma_{p^k-j} \subset \Sigma_{p^k}$.

3.12. Relation between Γ^q and Δ^q . There are isomorphisms of functors $U^q \approx U^{q+2k}$ and $V^q \approx V^{q+2k}$ for all $k \in \mathbb{Z}$, because E_* is an even periodic graded ring. The choice of such isomorphisms depends on a choice of isomorphism $E_* \approx E_{*+2k}$ of E_* -modules. Nonetheless, we obtain the following consequence.

3.13. Proposition. *There are isomorphisms $\Gamma^q \approx \Gamma^{q+2k}$ and $\Delta^q \approx \Delta^{q+2k}$ of graded rings under E_0 , for all $k \in \mathbb{Z}$.*

Proof. The only thing to note is that the isomorphisms $U^q \approx U^{q+2k}$ and $V^q \approx V^{q+2k}$ obtained from the periodicity of E actually give isomorphisms of underlying E_0 -modules, not merely of abelian groups. Thus the resulting ring isomorphisms are compatible with the inclusion of the subring E_0 . \square

Next, we will produce a chain of ring homomorphisms

$$\begin{array}{ccccc} \Gamma^{q+1} & & \Gamma^q & & \Gamma^{q-1} \\ f_{q+1} \downarrow & \nearrow g_{q+1} & f_q \downarrow & \nearrow g_q & \downarrow f_{q-1} \\ \Delta^{q+1} & & \Delta^q & & \Delta^{q-1} \end{array}$$

Any $\phi: I^q \rightarrow I^q$, applied to an augmented \mathbb{T} -algebra A , naturally induces an endomorphism $V^q \rightarrow V^q$ by passage to the indecomposables. Thus, we obtain a ring homomorphism $f_q: \Gamma^q \rightarrow \Delta^q$. Explicitly, the map $f_q[k]: \Gamma^q[k] \rightarrow \Delta^q[k]$ amounts to the natural map from the primitive subobject to the indecomposable quotient of $E_{-q}^\wedge B\Sigma_{p^k}^{-q\rho_{p^k}}$.

3.14. Proposition. *The map $f_q: \Gamma^q \rightarrow \Delta^q$ is an isomorphism if q is odd.*

Proof. The argument of the proof of [Rez09, Prop. 7.2] shows that $\mathbb{T}(E_* S^{-q}) \approx \bigoplus \pi_* L_{K(n)} \mathbb{P}_m \Sigma^{-q} E$ is, as a Hopf algebra, a primitively generated exterior algebra when q is odd. Thus, the evident map from primitives to indecomposables is an isomorphism. \square

Recall that the loop construction (§3.7) gives a functor $\Omega: \text{Alg}_{\mathbb{T}}^*/E_* \rightarrow (\text{Alg}_{\mathbb{T}}^*/E_*)_{\text{ab}} \subset \text{Alg}_{\mathbb{T}}^*/E_*$. Thus, there is a natural isomorphism of abelian groups $V^q(\Omega A) \approx I^q(A)$, and hence any endomorphism of V^q induces an endomorphism of I^q . We have thus defined a ring homomorphism $g_q: \Delta^q \rightarrow \Gamma^{q-1}$. Explicitly, the map $g_q[k]: \Delta^q[k] \rightarrow \Gamma^{q-1}[k]$ is induced by the “suspension” map

$$E_{-q}^\wedge B\Sigma_{p^k}^{-q\rho_{p^k}} \rightarrow E_{-q+1}^\wedge B\Sigma_{p^k}^{(-q+1)\rho_{p^k}},$$

which factors through the quotient $\Delta^q[k]$ and lands in the submodule $\Gamma^{q-1}[k]$.

3.15. Proposition. *The map $g_q: \Delta^q \rightarrow \Gamma^{q-1}$ is an isomorphism for all q .*

Proof. In light of (3.9) and (3.14), it is enough to show that the composite

$$\Delta^{2q}[k] \xrightarrow{g_{2q}} \Gamma^{2q-1}[k] \xrightarrow[\sim]{f_{2q-1}} \Delta^{2q-1}[k] \xrightarrow{g_{2q-1}} \Gamma^{2q-2}[k]$$

is an isomorphism. Because E_* is 2-periodic, it is enough to consider the case $q = 0$. Explicitly, we need to show that if we apply completed E_* -homology to the “zero-section” map

$$B\Sigma_{p^k}^+ \rightarrow B\Sigma_{p^k}^{2\bar{\rho}_{p^k}} \approx \Sigma^{-2} B\Sigma_{p^k}^{2\rho_{p^k}},$$

the image is exactly the submodule $\Gamma^{-2}[k]$. (Here $\bar{\rho}_{p^k}$ denotes the reduced real representation, so that $\rho_{p^k} = \mathbb{R} \oplus \bar{\rho}_{p^k}$.) That this is the case follows by Theorems 8.5 and 8.6 of [Str98], where the result is stated in “dual” form. Specifically, he proves that $\text{Prim} E^0 B\Sigma_{p^k} \rightarrow \text{Ind} E^0 B\Sigma_{p^k}$ is generated by the Euler class of $\bar{\rho}_{p^k} \otimes \mathbb{C} \approx 2\bar{\rho}_{p^k}$, where Prim is the kernel of restrictions, and Ind the quotient of transfers, along $\Sigma_i \times \Sigma_{p^k-i} \subset \Sigma_{p^k}$. \square

3.16. Corollary. *All of the rings Γ^q and Δ^q are isomorphic as graded rings under E_0 . Furthermore, each of the modules $\Gamma^q[k]$ and $\Delta^q[k]$ are finitely generated free E_0 -modules.*

Proof. By the above, we have isomorphisms $\Delta^0 \xrightarrow{g_0} \Gamma^{-1} \xrightarrow{f_{-1}} \Delta^{-1} \xrightarrow{g_{-1}} \Gamma^{-2}$, and the general isomorphism follows by (3.13). The freeness follows from (3.9). \square

For the remainder of this paper, we write Δ for the ring Δ^0 ; this is the ring we will show is Koszul.

4. KOSZUL RINGS

In this section, we develop the theory of Koszul rings in terms of the bar construction (following the original [Pri70]), and in the generality we need. Specifically, we describe the theory for a ring A which contains a commutative ring R , but which is *not central* in A , and we show that every such Koszul ring is a quadratic algebra. I believe these results are standard, but I do not know a convenient reference in the literature. In any case, we need to set up the interpretation of the Koszul property in terms of the bar construction, for our results in §7.

Furthermore, we will show that a ring A which is Koszul in our sense is necessarily *quadratic* (4.10). Once we show that the ring Δ is Koszul in our sense, we will have thus proved that it is quadratic.

In the following let $A = \bigoplus_{k \geq 0} A[k]$ be a graded associative ring, and suppose that $R = A[0]$ is commutative. It is important that we do *not* assume that R is central in A . We write $\epsilon: A \rightarrow R$ for the evident augmentation map.

4.1. Bar constructions for rings. Let M be a right A -module, and let N be a left A -module. The **two-sided bar construction** $\mathcal{B}(M, A, N)$ is the simplicial abelian group defined by

$$\mathcal{B}_q(M, A, N) = M \otimes_R \underbrace{A \otimes_R \cdots \otimes_R A}_{q \text{ copies}} \otimes_R N,$$

with face and boundary maps defined in the usual way.

Let $N\mathcal{B}(M, A, N)$ denote the *normalized* chain complex obtained from the bar resolution, (obtained by quotienting out by the image of degeneracy operators); we have $H_*N\mathcal{B}(M, A, N) = H_*\mathcal{B}(M, A, N)$.

Let $\overline{\mathcal{B}}(A) = \mathcal{B}(R, A, R)$, where R is viewed as a left or right A -module using the projection $\epsilon: A \rightarrow A[0] \approx R$ defined by $\epsilon(A[m]) = 0$ for $m > 0$. The complex $\overline{\mathcal{B}}(A)$ inherits a grading from the grading on A , so that there is an isomorphism of complexes $\overline{\mathcal{B}}(A) \approx \bigoplus \overline{\mathcal{B}}(A)[m]$, where

$$\overline{\mathcal{B}}_q(A)[m] \approx \bigoplus_{m_1 + \dots + m_q = m} A[m_1] \otimes_R \dots \otimes_R A[m_q],$$

where each index $m_i > 0$. Thus the homology of $\overline{\mathcal{B}}(A)$ is graded, too:

$$H_*\overline{\mathcal{B}}(A) \approx \bigoplus_{m \geq 0} H_*\overline{\mathcal{B}}(A)[m].$$

4.2. Proposition. *We have that*

$$H_0\overline{\mathcal{B}}(A) \approx A[0] = R,$$

and that

$$H_q\overline{\mathcal{B}}(A)[m] = 0 \quad \text{for } q > m.$$

Proof. The normalized complex $N\overline{\mathcal{B}}(A)$ is such that $N\overline{\mathcal{B}}(A)_q[m] = 0$ if $q > m$, or if $q = 0$ and $m > 0$. \square

Let $\mathcal{B}(A) \stackrel{\text{def}}{=} \mathcal{B}(A, A, A)$, the “big” two-sided bar construction on A . Since $\epsilon: A \rightarrow R$ is a map of A -bimodules, there is an induced surjective map $\mathcal{B}(A) \rightarrow \overline{\mathcal{B}}(A)$ of complexes. Let $\tilde{A} = \ker \epsilon$; this is an A -bimodule, so we can define complexes

$$\hat{\mathcal{B}}(A) \stackrel{\text{def}}{=} \mathcal{B}(A, A, \tilde{A}), \quad \check{\mathcal{B}}(A) \stackrel{\text{def}}{=} \mathcal{B}(\tilde{A}, A, A), \quad \ddot{\mathcal{B}}(A) \stackrel{\text{def}}{=} \mathcal{B}(\tilde{A}, A, \tilde{A}),$$

each of which is naturally a subcomplex of $\mathcal{B}(A)$.

4.3. Proposition. *The sequence of complexes*

$$0 \rightarrow \ddot{\mathcal{B}}(A) \xrightarrow{(incl., -incl.)} \hat{\mathcal{B}}(A) \oplus \check{\mathcal{B}}(A) \xrightarrow{(incl., incl.)} \mathcal{B}(A) \rightarrow \overline{\mathcal{B}}(A) \rightarrow 0$$

is exact.

Proof. In degree $q \geq 0$, $\mathcal{B}(A)_q$ has the form

$$\mathcal{B}(A)_q \approx \bigoplus_{m_0, \dots, m_{q+1} \geq 0} A[m_0] \otimes_R \dots \otimes_R A[m_{q+1}].$$

It is straightforward to identify the subgroups $\hat{\mathcal{B}}(A)_q$, $\check{\mathcal{B}}(A)_q$, and $\ddot{\mathcal{B}}(A)_q$ as consisting of those summands with either $m_{q+1} > 0$, $m_0 > 0$, or both, as the case may be. The result follows easily. \square

4.4. Definition of Koszul rings. We say that the graded ring A is **Koszul** if

$$H_q \overline{\mathcal{B}}(A)[m] = 0 \quad \text{for } q < m.$$

In view of the previous proposition, this means that if A is Koszul, then $\overline{\mathcal{B}}(A)[m]$ is a chain complex of R -modules whose homology is concentrated in the single dimension m .

4.5. Example. Let V be an R -bimodule. The **tensor algebra** TV is defined by

$$TV \stackrel{\text{def}}{=} \left(\bigoplus_{m \geq 0} \underbrace{V \otimes_R \cdots \otimes_R V}_{m \text{ copies}} \right).$$

It is a straightforward exercise to show that $H_q \overline{\mathcal{B}}(TV)[m] = 0$ for all $q \geq 0$ and all $m > 1$. Thus TV is Koszul.

For a Koszul ring A , let

$$C[m] \stackrel{\text{def}}{=} H_m \overline{\mathcal{B}}(A)[m].$$

4.6. Proposition. *Suppose that A is Koszul, and that each $A[m]$ is finitely generated and projective as a right R -module. Then each $C[m]$ is finitely generated and projective as a right R -module.*

Suppose each $A[m]$ is as above, and we write $\text{rank}_{\mathfrak{P}} A[m] = a_m$ for the rank of $A[m]_{\mathfrak{P}}$ as an $R_{\mathfrak{P}}$ -module for some prime $\mathfrak{P} \subset R$. Then each $C[m]$ has

$$\sum_{m=0}^{\infty} \text{rank}_{\mathfrak{P}} C[m] \cdot T^m = \left(\sum_{m=0}^{\infty} (-1)^m a_m \cdot T^m \right)^{-1}$$

in $\mathbb{Z}[[T]]$.

Proof. This is straightforward, using the fact that if P and Q are finitely generated and projective as right R -modules, then so is $P \otimes_R Q$, and that if R is a local ring, $\text{rank } P \otimes_R Q = (\text{rank } P)(\text{rank } Q)$. \square

4.7. Koszul resolutions. Let M be a right, and N a left R -module. Define a filtration $\{F_m\}$ of the normalized bar complex $NB(M, A, N)$, so that

$$F_m = F_m NB_q(M, A, N) \stackrel{\text{def}}{=} M \otimes_R \left(\bigoplus_{m_1 + \cdots + m_q \leq m} A[m_1] \otimes_R \cdots \otimes_R A[m_q] \right) \otimes_R N.$$

There are isomorphisms of complexes

$$F_m / F_{m-1} \approx M \otimes_R \overline{\mathcal{B}}(A)[m] \otimes_R N,$$

and thus a spectral sequence $E_1^{p,q} = H_p(M \otimes_R \overline{\mathcal{B}}(A)[q] \otimes_R N) \Rightarrow H_{p+q}(NB(M, A, N))$. This immediately gives the following.

4.8. Proposition. *Suppose that A is Koszul. If M and N are flat as right and left R -modules respectively, then the E_1 -term of the above spectral sequence collapses to a chain complex of the form*

$$\cdots \rightarrow M \otimes_R C[m] \otimes_R N \rightarrow M \otimes_R C[m-1] \otimes_R N \rightarrow \cdots.$$

This is the **Koszul complex** for computing $\text{Tor}^R(M, N)$.

4.9. Koszul rings are quadratic. A graded ring A is *quadratic* if it is generated as a ring under R by elements of degree one, and if all relations are generated by the homogeneous relations of degree 2. That is, A is quadratic if the natural map

$$T(A[1])/I \rightarrow A$$

is an isomorphism, where $T(V)$ represents the tensor algebra generated by an R -bimodule V , and I is a two-sided ideal generated by a sub-bimodule $P \subseteq A[1] \otimes_R A[1] \subset T(A[1])$.

4.10. Proposition. *If A is Koszul, and is flat as a right R -module, then A is quadratic.*

This result is well-known in the case that R is central; see, for instance, [PP05]. This will follow from the sharper result (4.12) below.

4.11. Proposition. *The ring A is generated by $A[1]$ if and only if $H_1\overline{\mathcal{B}}(A)[m] = 0$ for all $m > 1$.*

Proof. Since $\overline{\mathcal{B}}_0(A)[m] = 0$ for $m > 0$, we have for each $m \geq 2$ an exact sequence

$$\bigoplus_{\substack{k+\ell=m \\ k,\ell>0}} A[k] \otimes_R A[\ell] \xrightarrow{\text{mult}} A[m] \rightarrow H_1\overline{\mathcal{B}}(A)[m] \rightarrow 0.$$

Clearly, $H_1\overline{\mathcal{B}}(A)[m] = 0$ if and only if $A[m]$ is spanned by products of pairs of elements of strictly lower degree. The result follows. \square

4.12. Proposition. *Suppose A is flat as a right R -module. Then A is quadratic if and only if*

$$H_1\overline{\mathcal{B}}(A)[m] = 0 \quad \text{for all } m > 1,$$

and

$$H_2\overline{\mathcal{B}}(A)[m] = 0 \quad \text{for all } m > 2.$$

Proof. By the previous proposition, it suffices to show that if A is generated over R by $A[1]$, then A is quadratic if and only if $H_2\overline{\mathcal{B}}(A)[m] = 0$ for all $m > 2$.

Let $f: T(A[1]) \rightarrow A$ denote the homomorphism of rings which is the identity map on $A[1]$; it is surjective and grading preserving. Consider the resulting exact sequence of chain complexes

$$0 \rightarrow K[m] \xrightarrow{\gamma} \overline{\mathcal{B}}(TA[1])[m] \xrightarrow{g} \overline{\mathcal{B}}(A)[m] \rightarrow 0,$$

where g is induced by the algebra map f . Examining the exact sequence of complexes in degrees 1 and 2 gives the commutative diagram

$$\begin{array}{ccccc}
 & & 0 & & 0 \\
 & & \downarrow & & \downarrow \\
 \bigoplus_{\substack{p+q=m \\ p,q>0}} (K_1[p] \otimes_R A[1]^{\otimes q}) \oplus (A[1]^{\otimes p} \otimes_R K_1[q]) & \xrightarrow{\beta_m} & K_2[m] & \xrightarrow{\alpha_m} & K_1[m] \\
 & \nearrow \tilde{\beta} = \bigoplus (\gamma \otimes \text{id}, \text{id} \otimes \gamma) & \downarrow & & \downarrow \gamma \\
 & & \bigoplus_{\substack{p+q=m \\ p,q>0}} A[1]^{\otimes p} \otimes_R A[1]^{\otimes q} & \rightarrow & A[1]^{\otimes m} \\
 & & \downarrow g_2 = \bigoplus f \otimes f & & \downarrow f \\
 & & \bigoplus_{\substack{p+q=m \\ p,q>0}} A[p] \otimes_R A[q] & \longrightarrow & A[m] \\
 & & \downarrow & & \downarrow \\
 & & 0 & & 0
 \end{array}$$

in which the columns are exact. Since $g_2 \circ \tilde{\beta} = 0$, there is a unique lift β_m as shown in the diagram, and β_m must be surjective by the flatness hypothesis on A .

We have already observed that the complex $\overline{\mathcal{B}}(TA[1])[m]$ is acyclic for all $m > 1$ (4.5). Thus $H_2 \overline{\mathcal{B}}(A)[m] \approx H_1 K[m]$ for $m > 1$, and since $K_0[m] = 0$, we obtain an exact sequence

$$K_2[m] \xrightarrow{\alpha} K_1[m] \rightarrow H_2 \overline{\mathcal{B}}(A)[m] \rightarrow 0.$$

Putting this all together, we find that (for $m > 2$), $H_2 \overline{\mathcal{B}}(A)[m] = 0$ if and only if α_m is surjective, if and only if $\alpha_m \beta_m$ is surjective. But $\alpha_m \beta_m$ being surjective means exactly that $K_1[m]$ (the module of relations of A of degree m) is generated by relations of lower degree. \square

5. LINEARIZATION OF FUNCTORS

In this section, we discuss a certain “linearization” operation on functors between abelian categories. The linearization construction gives us a formal approach to the algebra Δ of §3, and will be used later to obtain a bar-resolution of Δ from partition complexes.

5.1. The linearization construction. Let \mathcal{A} be an additive category, and let \mathcal{B} be an abelian category. Let $F: \mathcal{A} \rightarrow \mathcal{B}$ be a (not necessarily additive) functor. We say F is **reduced** if $F0 \approx 0$.

For such a reduced functor F , we define $\mathcal{L}_F: \mathcal{A} \rightarrow \mathcal{B}$ together with a natural transformation $\epsilon: F \rightarrow \mathcal{L}_F$ by setting $\mathcal{L}_F(X)$ to be the coequalizer of

$$F(X \oplus X) \begin{array}{c} \xrightarrow{F(p_1+p_2)} \\ \xrightarrow{F(p_1)+F(p_2)} \end{array} F(X)$$

where $p_i: X \oplus X \rightarrow X$ for $i = 1, 2$ are the two projections.

5.2. Proposition. *The functor \mathcal{L}_F is additive; any natural map $F \rightarrow G$ to an additive functor G factors uniquely through $\epsilon: F \rightarrow \mathcal{L}_F$.*

Proof. For the first part, note that if $f, g: Y \rightarrow X$ are two maps in \mathcal{A} , then

$$\epsilon F(f + g) = \epsilon F(p_1 + p_2)F((f, g)) = \epsilon(F(p_1) + F(p_2))F((f, g)) = \epsilon(F(f) + F(g)),$$

as maps $F(Y) \rightarrow \mathcal{L}_F(X)$, from which it follows that $\mathcal{L}_F(f + g) = \mathcal{L}_F(f) + \mathcal{L}_F(g)$. The second part follows from the observation that $\mathcal{L}_G \approx G$ when G is additive. \square

Let $\perp F(X) \stackrel{\text{def}}{=} \ker \left[F(X \oplus X) \xrightarrow{(F(p_1), F(p_2))} F(X) \oplus F(X) \right]$, and write $\beta_F: \perp F(X) \rightarrow F(X \oplus X)$ for the inclusion. Then there is an exact sequence

$$\perp F(X) \xrightarrow{\gamma_F} F(X) \xrightarrow{\epsilon} \mathcal{L}_F(X) \rightarrow 0.$$

where $\gamma_F = F(p_1 + p_2) \circ \beta_F$.

5.3. A “chain rule”. Given functors $F: \mathcal{A} \rightarrow \mathcal{B}$ and $G: \mathcal{B} \rightarrow \mathcal{C}$, where \mathcal{A} is additive and \mathcal{B} and \mathcal{C} are abelian, there is a unique natural transformation $c: \mathcal{L}_{F \circ G} \rightarrow \mathcal{L}_F \circ \mathcal{L}_G$ such that $c\epsilon_{F \circ G} = \epsilon_F \circ \epsilon_G: F \circ G \rightarrow \mathcal{L}_F \circ \mathcal{L}_G$, since $\mathcal{L}_F \circ \mathcal{L}_G$ is additive. Our chain rule says that the transformation c is an isomorphism whenever things split.

5.4. Proposition. *Let X be an object of \mathcal{A} . Suppose that there are direct sum decompositions*

$$A \oplus B \xrightarrow[\sim]{(i_A, i_B)} G(X), \quad B \oplus C \xrightarrow[\sim]{(i'_B, i'_C)} \perp G(X),$$

such that $\epsilon_G i_A: A \rightarrow \mathcal{L}_G(X)$ is an isomorphism, $\gamma_G i'_B = i_B$, and $\gamma_G i'_C = 0 = \epsilon_G i'_B$. Then the natural map $c: \mathcal{L}_{F \circ G}(X) \rightarrow \mathcal{L}_F(\mathcal{L}_G(X))$ is an isomorphism.

5.5. Corollary. *Let $F, G: \mathcal{A} \rightarrow \mathcal{A}$ be functors on an abelian category \mathcal{A} , and suppose that $G(X)$ and $\mathcal{L}_G(X)$ are projective whenever X is a projective object. Then $c: \mathcal{L}_{F \circ G}(X) \rightarrow \mathcal{L}_F(\mathcal{L}_G(X))$ is an isomorphism for all projective X .*

The proof of (5.4) is given at the end of this section.

5.6. The ring Δ is the linearization of the monad $\tilde{\mathbb{T}}$.

5.7. Proposition. *Let $m = p^k$, and let M be a free E_* -module concentrated in even degree. Then the natural map $\mathcal{L}_{\mathbb{T}\langle m \rangle}(M) \rightarrow \Delta[k] \otimes_{E_0} M$ is an isomorphism. If $m \neq p^k$, $\mathcal{L}_{\mathbb{T}\langle m \rangle}(M) = 0$. These maps fit together to give an isomorphism $\mathcal{L}_{\tilde{\mathbb{T}}}(M) \approx \Delta \otimes_{E_0} M$.*

Proof. Since \mathbb{T} commutes with filtered colimits (§3.2(6)), so does $\mathcal{L}_{\mathbb{T}\langle m \rangle}$, and so it is enough to consider finite free modules $M = (E_*)^k$. Since both $\mathcal{L}_{\mathbb{T}\langle m \rangle}$ and $\Delta \otimes_{E_*} (-)$ are additive, it is enough to consider the module $M = E_*$. Now we compute that for $m \geq 1$,

$$\begin{aligned} \perp \mathbb{T}\langle m \rangle(E_*) &= \text{Ker} \left[\mathbb{T}\langle m \rangle(E_* \oplus E_*) \xrightarrow{(\mathbb{T}\langle m \rangle p_1, \mathbb{T}\langle m \rangle p_2)} \mathbb{T}\langle m \rangle(E_*) \oplus \mathbb{T}\langle m \rangle(E_*) \right] \\ &\approx \bigoplus_{0 < i < m} E_* B(\Sigma_i \times \Sigma_{m-i}). \end{aligned}$$

The map $\mathbb{T}\langle m \rangle(p_1 + p_2): \mathbb{T}\langle m \rangle(E_* \oplus E_*) \rightarrow \mathbb{T}\langle m \rangle(E_*)$ computes the effect of $\mathbb{P}_m(p_1 + p_2): \mathbb{P}_m(E \vee E) \rightarrow \mathbb{P}_m(E)$ on homotopy, and we see that $\mathbb{T}\langle m \rangle(E_*)$ is the cokernel of $\bigoplus_{0 < i < m} E_* B(\Sigma_i \times \Sigma_{m-i}) \rightarrow E_* B\Sigma_m$ induced by transfers. This map is surjective if m is not a power of p ; if $m = p^k$ then in degree 0 the cokernel is precisely $\Delta[k]$, and in general degree the cokernel is $\Delta[k] \otimes_{E_0} E_*$. \square

In particular, it follows that $\mathcal{L}_{\tilde{\mathbb{T}}}(M)$ is a free E_* -module whenever M is, since Δ is a free E_0 -module (3.16).

5.8. Proposition. *There is an isomorphism*

$$\mathcal{L}_{\mathcal{B}(\tilde{\mathbb{T}})}(M) \approx \mathcal{B}(\Delta) \otimes_{E_0} M$$

of simplicial E_ -modules, which is compatible with grading, so that $\mathcal{L}_{\mathcal{B}(\tilde{\mathbb{T}})\langle m \rangle}(M) \approx \mathcal{B}(\Delta)[k] \otimes_{E_0} M$, where $m = p^k$.*

Proof. If M is a free module, then $\tilde{\mathbb{T}}(M)$ and $\mathcal{L}_{\tilde{\mathbb{T}}(M)}$ are free. Thus, the chain rule (5.5) applies to show that $\mathcal{L}_{\tilde{\mathbb{T}}^{\circ q}}(M) \approx \mathcal{L}_{\tilde{\mathbb{T}}}^{\circ q}(M)$ is an isomorphism for $q \geq 0$. The result follows using (5.7). \square

We can make a more refined statement about this isomorphism: all contributions to $\mathcal{L}_{\mathcal{B}(\tilde{\mathbb{T}})}$ come from the pure weight part (§2.2) of $\tilde{\mathbb{T}}^{\circ q}\langle m \rangle$.

5.9. Proposition. *For all $m = m_1 \cdots m_q$ with $m_i = p^{k_i}$ and $k_i \geq 0$, the diagram*

$$\begin{array}{ccc} \mathcal{L}_{\mathbb{T}\langle m_1 \rangle \circ \cdots \circ \mathbb{T}\langle m_q \rangle} M & \longrightarrow & \Delta[k_1] \otimes_{E_0} \cdots \otimes_{E_0} \Delta[k_q] \otimes_{E_0} M \\ \downarrow & & \downarrow \\ \mathcal{L}_{\tilde{\mathbb{T}}^{\circ q}}(M) & \longrightarrow & \Delta \otimes_{E_0} \cdots \otimes_{E_0} \Delta \otimes_{E_0} M \end{array}$$

commutes.

Proof. This is straightforward from the naturality of the linearization construction. \square

5.10. Proof of the chain rule. We give here a tedious elementary proof of (5.4); afterwards, we indicate how a somewhat more conceptual proof may be constructed, using the results of [JM04].

Proof. First we claim that $c: \mathcal{L}_{F \circ G}(X) \rightarrow \mathcal{L}_F(\mathcal{L}_G(X))$ is an epimorphism. The commutative diagram

$$\begin{array}{ccccc} F(A) & \xrightarrow{F(i_A)} & F(G(X)) & \xrightarrow{\epsilon_{F \circ G}} & \mathcal{L}_{F \circ G}(X) \\ & \searrow \sim & \downarrow F(\epsilon_G) & & \downarrow c \\ & F(\epsilon_G i_A) & F(\mathcal{L}_G(X)) & \xrightarrow{\epsilon_F} & \mathcal{L}_F(\mathcal{L}_G(X)) \end{array}$$

shows that c factors through an epimorphism $F(A) \rightarrow \mathcal{L}_F(\mathcal{L}_G(X))$, and thus is an epimorphism.

Let $D = \text{Ker} \left[F(A \oplus B) \xrightarrow{(F(p_1), F(p_2))} F(A) \times F(B) \right]$, and write $\alpha: D \rightarrow F(A \oplus B)$ for the inclusion. Let $i_D: D \rightarrow F(G(X))$ be defined by $F((i_A, i_B)) \circ \alpha$. The commutative diagram

$$\begin{array}{ccccc} F(A) \oplus F(B) \oplus D & \xrightarrow[\sim]{(Fi_A, Fi_B, i_D)} & F(G(X)) & \xrightarrow{\epsilon_{F \circ G}} & \mathcal{L}_{F \circ G}(X) \\ p_1 \downarrow & & F\epsilon_G \downarrow & & \downarrow c \\ F(A) & \xrightarrow[\sim]{F(\epsilon_G i_A)} & F(\mathcal{L}_G(X)) & \xrightarrow{\epsilon_F} & \mathcal{L}_F(\mathcal{L}_G(X)) \end{array}$$

shows that we can identify the kernel of the projection $F(G(X)) \rightarrow \mathcal{L}_F(\mathcal{L}_G(X))$ with the image of the bottom horizontal map in

$$\begin{array}{ccc} & & F(G(X \oplus X)) \\ & \nearrow (g_1, g_2, g_3) & \downarrow F(G(p_1 + p_2)) - F(G(p_1)) - F(G(p_2)) \\ \perp F(A) \oplus F(B) \oplus D & \xrightarrow{((Fi_A)\gamma_F, Fi_B, i_D)} & F(G(X)) \end{array}$$

We will construct a section (g_1, g_2, g_3) making the above diagram commute, and thus prove that the projection $F(G(X)) \rightarrow \mathcal{L}_F(\mathcal{L}_G(X))$ factors through an isomorphism $c: \mathcal{L}_{F \circ G}(X) \rightarrow \mathcal{L}_F(\mathcal{L}_G(X))$, as desired.

Let $g_1: \perp F(A) \rightarrow F(G(X \oplus X))$ be defined by the composite

$$\perp F(A) \xrightarrow{\beta_F} F(A \oplus A) \xrightarrow{F(i_A \oplus i_A)} F(G(X) \oplus G(X)) \xrightarrow{F((G(i_1), G(i_2)))} F(G(X \oplus X)).$$

It is straightforward to check that

$$\begin{aligned} F(G(p_\alpha)) \circ F((G(i_1), G(i_2))) \circ F(i_A \oplus i_A) &= F(i_A) \circ F(p_\alpha), \quad (\alpha = 1, 2) \\ F(G(p_1 + p_2)) \circ F((G(i_1), G(i_2))) \circ F(i_A \oplus i_A) &= F(i_A) \circ F(p_1 + p_2). \end{aligned}$$

Thus

$$(F(G(p_1 + p_2)) - F(G(p_1)) - F(G(p_2))) \circ g_1 = F(i_A) \circ \gamma_F$$

as desired.

Let $g_2: F(B) \rightarrow F(G(X \oplus X))$ be defined by the composite

$$F(B) \xrightarrow{F(i'_B)} F(\perp G(X)) \xrightarrow{F\beta_G} F(G(X \oplus X)).$$

Since

$$F(G(p_\alpha)) \circ F(\beta_G) = 0, \quad (\alpha = 1, 2), \quad F(G(p_1 + p_2)) \circ F(\beta_G) = F(\gamma_G),$$

we see that

$$(F(G(p_1 + p_2)) - F(G(p_1)) - F(G(p_2))) \circ g_2 = F(\gamma_G) \circ F(i'_B) = F(i_B),$$

as desired.

Let $g_3: D \rightarrow F(G(X \oplus X))$ be defined by the composite

$$D \xrightarrow{\alpha} F(A \oplus B) \xrightarrow{F(i_A \oplus i_B)} F(G(X) \oplus G(X)) \xrightarrow{F((G(i_1), G(i_2)))} F(G(X \oplus X)).$$

It is straightforward to check that

$$\begin{aligned} F(G(p_1)) \circ F((G(i_1), G(i_2))) \circ F(i_A \oplus i_B) &= F(i_A) \circ F(p_1), \\ F(G(p_2)) \circ F((G(i_1), G(i_2))) \circ F(i_A \oplus i_B) &= F(i_B) \circ F(p_2), \\ F(G(p_1 + p_2)) \circ F((G(i_1), G(i_2))) \circ F(i_A \oplus i_B) &= F((i_A, i_B)). \end{aligned}$$

Thus

$$(F(G(p_1 + p_2)) - F(G(p_1)) - F(G(p_2))) \circ g_3 = F(i_D),$$

as desired. \square

5.11. Proof of the chain rule, using Johnson-McCarthy. We briefly describe how one may produce a proof of the chain-rule using the work of [JM04]. In that paper, the authors describe a “derived linearization” construction which, given a functor $F: \mathcal{A} \rightarrow \mathcal{B}$ from an additive category to an abelian category such that $F0 = 0$, produces a functor $D_1F: \mathcal{A} \rightarrow \text{Ch}\mathcal{B}$ to the category of chain complexes in \mathcal{B} . In degrees 1 and 0, the chain complex D_1F has the form

$$\cdots \rightarrow \perp F \xrightarrow{\gamma_F} F,$$

and thus $\mathcal{L}_F = H_0D_1F$.

According to [JM04, Lemma 5.7], there is a quasi-isomorphism $D_1F \circ D_1G \approx D_1(F \circ G)$, where the left-hand side is the total complex of the bicomplex obtained by applying D_1F degreewise to D_1G . Thus, to recover (5.4) from their result, it is necessary to show that $H_0(D_1F \circ D_1G) \approx H_0D_1F \circ H_0D_1G$. This is where the hypotheses on $G(X)$ come into play. The functor D_1F is additive up to quasi-isomorphism (i.e., $D_1F(X) \oplus D_1F(Y) \rightarrow D_1F(X \oplus Y)$ is always a quasi-isomorphism). Thus, under the hypotheses of (5.4), applying D_1F to the sequence $\perp G(X) \rightarrow G(X)$ gives a map quasi-isomorphic to

$$D_1F(B) \oplus D_1F(C) \xrightarrow{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}} D_1F(A) \oplus D_1F(B),$$

and thus $H_0(D_1F \circ D_1G(X)) \approx H_0D_1F(A) \approx H_0D_1F \circ H_0D_1G(X)$, as desired.

6. POSETS AND THE PARTITION COMPLEX

In this section, we describe the *partition complex* of a finite set, and use it to define what we call the *uniform complex* U_m . Both complexes play a crucial role in the rest of the paper.

6.1. Notation for posets and their nerves. Let X be a poset. The simplicial nerve functor gives a fully faithful embedding of the category of posets into the category of simplicial sets. Thus, in this paper, it will be convenient to regard a poset as merely a certain kind of simplicial set, and so we use the same notation of a poset and its simplicial nerve; an *element* of a poset is exactly a *vertex* of the simplicial set. If $x_0 \leq x_1 \leq \cdots \leq x_q$ is a finite increasing sequence of elements of X , we write $[x] = [x_0 \leq x_1 \leq \cdots \leq x_q]$ for the corresponding q -simplex of the nerve. Non-degenerate simplices of the nerve correspond to chains in which each inequality is strict.

Suppose now that the poset X has maximal and minimal elements, which we denote $\underline{1}$ and $\underline{0}$ respectively. In what follows we will always assume $\underline{1} \neq \underline{0}$. We introduce the following notation:

- (1) Let \hat{X} denote the maximal subposet of X which does not include $\underline{0}$; we also use this notation to denote the nerve of this poset.
- (2) Let \check{X} denote the maximal subposet of X which does not include $\underline{1}$; we also use this notation to denote the nerve of this poset.
- (3) Let $\ddot{X} = \hat{X} \cap \check{X}$ as posets; the nerve of \ddot{X} is also an intersection of nerves.
- (4) Let X^\diamond denote the sub-simplicial set of X defined by $\hat{X} \cup \check{X}$.
- (5) Let \overline{X} denote the pointed simplicial set defined by X/X^\diamond .

Note that neither X^\diamond nor \overline{X} are generally nerves of posets.

Observe that since the simplicial sets \hat{X} and \check{X} are contractible, we have that X^\diamond is weakly equivalent to the unreduced suspension of \ddot{X} , and \overline{X} is weakly equivalent to the reduced suspension of X^\diamond .

The **order complex** of a poset X , as studied in algebraic combinatorics, is the simplicial complex whose q -simplices are chains $x_0 \not\leq x_1 \not\leq \cdots \not\leq x_q$ of elements in X . It corresponds precisely to the simplicial set \ddot{X} .

Say that a q -simplex $[x_0 \leq \cdots \leq x_q]$ of X is **essential** if it is non-degenerate and not contained in the subcomplex X^\diamond . An essential q -simplex is precisely a chain of the form $[\underline{0} = x_0 \not\leq x_1 \not\leq \cdots \not\leq x_{q-1} \not\leq x_q = \underline{1}]$. The essential q -simplices of X are exactly the non-basepoint non-degenerate simplices of the quotient \overline{X} . For $q > 1$, there is a bijective correspondence between *essential q -simplices of X* and *non-degenerate $(q-2)$ -simplices of \ddot{X}* , relating $[\underline{0} = x_0 \not\leq \cdots \not\leq x_q = \underline{1}]$ and $[x_1 \not\leq \cdots \not\leq x_{q-1}]$; in addition, there is a unique essential 1-simplex $[\underline{0} \not\leq \underline{1}]$ of X . In other words, as a combinatorial object, \overline{X} contains the same information as the order complex of X .

6.2. Partition complex. Fix $m \geq 0$. Let $P = P_m$ denote the poset of equivalence relations on the set $\underline{m} = \{1, \dots, m\}$, ordered by refinement. Thus, a q -simplex of P is a chain $[E_0 \leq \cdots \leq E_q]$ of equivalence relations, where we write $E \leq E'$ if E is “finer” than E' , i.e., if $x \sim_E y$ implies $x \sim_{E'} y$ for all x and y in the set.

There is an evident action of the symmetric group Σ_m on P_m , obtained from the action of Σ_m on \underline{m} .

6.3. Transitive abelian subgroups of Σ_m . Recall that the symmetric group Σ_m comes with a tautological action on the finite set $\underline{m} = \{1, \dots, m\}$. We say that a subgroup $H \subseteq \Sigma_m$ is **transitive** if it acts transitively on \underline{m} via the tautological action. In this paper we will have a lot to say about transitive *abelian* subgroups of Σ_m . The following result is standard and easy.

6.4. Proposition. *Let $m \geq 1$.*

- (1) *Every transitive abelian subgroup of Σ_m has order m .*
- (2) *Every abelian group of order m is isomorphic to some transitive abelian subgroup of Σ_m .*
- (3) *Two transitive abelian subgroups A and B of Σ_m are conjugate if and only if they are isomorphic as abstract abelian groups.*
- (4) *The normalizer of a transitive abelian subgroup A of Σ_m is isomorphic to $A \rtimes \text{Aut}(A)$, where $\text{Aut}(A)$ is the automorphism group of the abelian group A .*

- (5) *The transitive abelian subgroups of Σ_m are precisely the maximal abelian subgroups; i.e., each transitive abelian subgroup is its own centralizer in Σ_m .*

6.5. Fixed subcomplexes of the partition complex. Fix a transitive abelian subgroup A in Σ_m . Given a subgroup $V \subseteq A$, we define an equivalence relation on E_V on \underline{m} by $x \sim_{E_V} y$ if and only if $vx = y$ for some $v \in A$.

6.6. Proposition. *Let A be a transitive abelian subgroup of Σ_m . An equivalence relation E on \underline{m} is fixed by the action of A if and only if $E = E_V$ for some subgroup V of A .*

6.7. Corollary. *Let A be a transitive abelian subgroup of Σ_m . The function $V \mapsto E_V$ defines an isomorphism between the poset of subgroups of A and the poset P_m^A . Under this correspondence, $0 \mapsto E_0 = \underline{0}$ and $A \mapsto E_A = \underline{1}$.*

6.8. Complete filtrations. Let $m = p^k$ for some $k \geq 0$. A **pure partition** of a set is an equivalence relation E such that all the equivalence classes of E have the same size. A partition E of \underline{m} is pure if and only if all the equivalence classes have order p^j , for some fixed $j \in \{0, \dots, k\}$. We say that the **mesh** of such a pure partition is j ; we write $\text{mesh}(E) = j$.

Thus, if $A \subseteq \Sigma_m$ is a transitive abelian subgroup and $V \subseteq A$ is a subgroup, then E_V is a pure partition with $\text{mesh}(E_V) = \text{length}(V)$.

A **complete filtration** of \underline{m} is a chain $E_0 \not\subseteq E_1 \not\subseteq \dots \not\subseteq E_k$ of pure partitions of \underline{m} ; thus $\text{mesh}(E_j) = j$. Let $CF = CF_m$ denote the set of complete filtrations in \underline{m} . Then, CF_m is a subset of the set of essential k -simplices of P_m . As a set with a Σ_m action, we see that

$$CF_m \approx \Sigma_m / \underbrace{\Sigma_p \wr \dots \wr \Sigma_p}_{k \text{ times}}.$$

If $A \subseteq \Sigma_m$ is a transitive abelian subgroup, then CF_m^A is in natural bijective correspondence with the set of complete filtrations $0 = V_0 \subsetneq V_1 \subsetneq \dots \subsetneq V_k = A$ of the group A .

6.9. The uniform complex. Let $m = p^k$. Consider the simplicial set $P \times CF = P_m \times CF_m$, where the set CF is regarded as a discrete simplicial set. For a transitive abelian subgroup $A \subset \Sigma_m$, let

$$U^A = U_m^A \stackrel{\text{def}}{=} P^A \times CF^A,$$

and let $U = U_m = \bigcup U^A$ where the union is taken over all transitive abelian subgroups. Thus U is a sub-simplicial set of $P \times CF$. Let

$$U^{\diamond A} \stackrel{\text{def}}{=} P^{\diamond A} \times CF^A, \quad \text{and} \quad \overline{U}^A \stackrel{\text{def}}{=} U^A / U^{\diamond A} \approx \overline{P}^A \wedge CF_+^A,$$

and let $U^\diamond = \bigcup U^{\diamond A}$ and let $\overline{U} = U / U^\diamond$. We can similarly define \hat{U} , \check{U} , \ddot{U} , etc. We call U the **uniform complex**. It carries an action of Σ_m inherited from the action on $P \times CF$; an element $\sigma \in \Sigma_m$ carries U^A into $U^{\sigma A \sigma^{-1}}$.

A q -simplex of U^A is a pair $([W], [V])$, where $[0 = V_0 \subsetneq \dots \subsetneq V_k = A]$ is a complete chain of subgroups of A , and $[W_0 \subseteq \dots \subseteq W_q]$ is a chain of subgroups in A . We say that this is an **essential** q -simplex if $[W]$ is an essential q -simplex of P , that is, if $0 = W_0 \subsetneq W_1 \dots \subsetneq W_{q-1} \subsetneq W_q = A$.

6.10. Remark. In light of what we prove in the next section (e.g., (7.15)), one might think it would be enough to restrict to the subcomplex $P'_m \subset P_m$ consisting of simplices whose Σ_m -isotropy groups act transitively on \underline{m} . However, to prove the homological vanishing result

we will need, we would like use a transfer argument to reduce it to an analogous statement about $P_m \times CF_m$. The issue is that the subcomplex of $P_m \times CF_m$ consisting of simplices with Σ_m -transitive isotropy is a *proper* subcomplex of the product $P'_m \times CF_m$. This proper subcomplex contains the uniform complex U_m .

6.11. *Remark.* The following seems to me to be true: U is the union of the U^A , where A ranges over the *elementary abelian* transitive subgroups of Σ_m . Unfortunately, I have been unable to construct a proof of this fact. (Compare with the fact that the union of P^A s over transitive abelian A is the same as the union over transitive elementary abelian A , which is not too hard to prove.) Luckily, it turns out that the answer to this question is not needed for the proof of the main theorem of this paper.

7. THE RELATION BETWEEN STANDARD RESOLUTIONS

This section contains the key observation of this paper: that the bar resolution of Δ , which we need to understand in order to show that Δ is Koszul, is a linearization of the bar resolution of \mathbb{T} , which can be expressed in terms of partition complexes.

7.1. **Transitive E -homology.** Let $S \subset \underline{2}^{\underline{m}}$ denote the set of surjective functions $\underline{m} \rightarrow \underline{2}$, equipped with the evident Σ_m -action.

For a Σ_m -space X , define

$$Q(X) \stackrel{\text{def}}{=} \text{Cok}[E_*^\wedge(X \times S)_{h\Sigma_m} \rightarrow E_*^\wedge X_{h\Sigma_m}],$$

where the map is the one induced by projection $\pi: X \times S \rightarrow X$. For lack of a better name, we will call this the **transitive E -homology** of the Σ_m -space X . The functor Q is not a homology theory; however, it does take finite coproducts of Σ_m -spaces to direct sums. The name comes from the fact that Q only sees Σ_m -orbits with transitive isotropy, according to the following proposition.

7.2. **Lemma.** *Suppose that X is a finite Σ_m -set such that for all $x \in X$, the isotropy group $H \subseteq \Sigma_m$ of x does not act transitively on \underline{m} . Then $Q(X) = 0$.*

Proof. Since Q preserves finite coproducts, it suffices to consider $X = \Sigma_m/H$. If H does not act transitively on \underline{m} , there exists a surjective function $f: \underline{m} \rightarrow \underline{2}$ which is preserved by the H -action, and thus $\sigma H \mapsto (\sigma H, f\sigma^{-1}): X \rightarrow X \times S$ is a Σ_m -equivariant section of the projection $X \times S \rightarrow X$. \square

If X is a pointed Σ_m -space, we write $\tilde{Q}(X)$ for the “reduced” transitive E -homology, which is defined to be the cokernel of $E_*^\wedge(X \wedge S_+)_{h\Sigma_m} \rightarrow E_*^\wedge(X_{h\Sigma_m})$, (where homotopy orbits are computed in the category of pointed spaces.)

In the remainder of this paper, we will apply Q and \tilde{Q} to *discrete* Σ_m -spaces. Typically, we will have a Σ_m -equivariant simplicial set X , and we will apply Q to each simplicial degree of X separately; thus, if X is a pointed Σ_m -equivariant simplicial set, $\tilde{Q}(X)$ denotes the simplicial abelian group with

$$\tilde{Q}(X)_q \stackrel{\text{def}}{=} \tilde{Q}(X_q).$$

7.3. Transitive E -homology commutes with transfers. To any map $f: X \rightarrow Y$ of finite Σ_m -sets, there is a transfer map $f_!: \Sigma_+^\infty Y_{h\Sigma_m} \rightarrow \Sigma_+^\infty X_{h\Sigma_m}$ in the homotopy category of spectra.

7.4. Proposition. *If $f: X \rightarrow Y$ is a map of finite Σ_m -sets, then the map $f_!: E_*^\wedge(Y_{h\Sigma_m}) \rightarrow E_*^\wedge(X_{h\Sigma_m})$ induced by transfer factors through a unique map $f_!: Q(Y) \rightarrow Q(X)$.*

Proof. Immediate from the identity $\Sigma_+^\infty \pi \circ (f_! \wedge \text{id}_{\Sigma_+^\infty S}) = f_! \circ \Sigma_+^\infty \pi$. \square

7.5. Corollary. *Let $H \subseteq \Sigma_m$ be a subgroup of index prime to p . Then there exist maps*

$$Q(X) \xrightarrow{i} Q(X \times \Sigma_m/H) \xrightarrow{j} Q(X),$$

natural in the Σ_m -set X , such that $ji = \text{id}$.

Proof. If $q: \Sigma_m/H \rightarrow *$ denotes the projection, it is standard that the composite $g = (\Sigma_+^\infty q) \circ q_!: S^0 \rightarrow S^0$ is a p -local equivalence. Thus we let $j = Q(\text{id}_X \times q)$, and $i = (\text{id}_X \times q)_! \circ (Q(g))^{-1}$. \square

7.6. Transitive E -homology as a linearization. Fix a space W equipped with a Σ_m -action. Let $F: h\text{Spectra} \rightarrow \text{Mod}_{E_*}$ be the functor defined by

$$F(Y) \stackrel{\text{def}}{=} E_*^\wedge(\Sigma_+^\infty W \wedge Y^{\wedge m})_{h\Sigma_m}.$$

Let $\mathcal{L}_F: h\text{Spectra} \rightarrow \text{Mod}_{E_*}$ be the linearization of F , as described in §5.

7.7. Proposition. *For all spaces X there is a natural isomorphism*

$$\mathcal{L}_F(\Sigma_+^\infty X) \rightarrow Q(W \times X^m).$$

Proof. The sequence

$$Y^{\wedge m} \wedge \Sigma_+^\infty S \rightarrow (Y \vee Y)^{\wedge m} \xrightarrow{(p_1, p_2)} Y^{\wedge m} \vee Y^{\wedge m}$$

is a split cofibration sequence in the category of Σ_m -spectra, from which it follows that the map $\gamma_F: \perp F(Y) \rightarrow F(Y)$ is isomorphic to the map $E_*^\wedge(\Sigma_+^\infty W \wedge Y^{\wedge m} \wedge \Sigma_+^\infty S)_{h\Sigma_m} \rightarrow E_*^\wedge(\Sigma_+^\infty W \wedge Y^{\wedge m})_{h\Sigma_m}$ induced by projection $S \rightarrow *$. \square

7.8. Transitive E -homology of the partition complex. We have the following.

7.9. Proposition. *Let $P = P_m$ be the nerve of the partition poset. Then the sequence of simplicial abelian groups*

$$0 \rightarrow Q(\ddot{P}) \xrightarrow{(\text{incl.}, -\text{incl.})} Q(\hat{P}) \oplus Q(\check{P}) \xrightarrow{(\text{incl.}, \text{incl.})} Q(P) \rightarrow \tilde{Q}(\overline{P}) \rightarrow 0$$

is exact.

Proof. For $q \geq 0$, let P_q denote the q -simplices of P . Then

$$P_q \approx (P_q - P_q^\diamond) \amalg (\hat{P}_q - \check{P}_q) \amalg (\check{P}_q - \ddot{P}_q) \amalg \ddot{P}_q$$

as Σ_m -sets, and the result follows because Q preserves coproducts. \square

7.10. The partition complex and E_∞ -operads. A **symmetric sequence** is a functor $A: \Sigma \rightarrow \text{Spaces}$, where, Σ denotes the groupoid of finite sets and isomorphisms. A symmetric sequence A determines a functor

$$\mathcal{C}_A: \text{Spaces} \rightarrow \text{Spaces} \quad \text{by} \quad \mathcal{C}_A(X) \stackrel{\text{def}}{=} \coprod_{m \geq 0} (A(\underline{m}) \times X^m)_{\Sigma_m},$$

and the assignment $A \mapsto \mathcal{C}_A$ is functorial. There is a monoidal product $A, B \mapsto A \circ B$, which has the property that $\mathcal{C}_{A \circ B} \approx \mathcal{C}_A \mathcal{C}_B$. This monoidal product satisfies the formula

$$(A \circ B)(S) \approx \coprod_{n, f: S \rightarrow \underline{n}} A(\underline{n}) \times \prod_{s \in S} B(f^{-1}(s))$$

where the coproduct runs over integers $n \geq 0$ and functions $f: S \rightarrow \underline{n}$. An operad is a monoid with respect to this monoidal product.

If O is an operad, then $\mathcal{B}(O) = \mathcal{B}(O, O, O)$ is a simplicial object in symmetric sequences.

7.11. Proposition. *Let O be the non-unital E_∞ -operad in spaces. For each $m \geq 0$ there is a map $\mathcal{B}(O)\langle m \rangle \rightarrow P_m$ of simplicial Σ_m -spaces, which is a weak equivalence of spaces in each simplicial degree. That is, $O^{\circ(q+2)}\langle m \rangle \rightarrow (P_m)_q$ is a weak equivalence.*

Proof. A standard and well-known combinatorial argument. In fact, taking O to be the non-unital commutative operad, with $O(S)$ a one-point space for all non-empty S , gives an isomorphism $\mathcal{B}(O)\langle m \rangle \approx P_m$ of simplicial Σ_m -spaces. \square

We may consider the monad $\tilde{C} = C_O$ associated to O , together with its associated monadic bar construction $\mathcal{B}(\tilde{C}) = \mathcal{B}(\tilde{C}, \tilde{C}, \tilde{C})$. Evaluating at a space X gives a simplicial space $\mathcal{B}(\tilde{C})(X) = \mathcal{B}(\tilde{C}, \tilde{C}, \tilde{C}(X))$, and applying (7.11) leads to the following.

7.12. Proposition. *There is an isomorphism*

$$\mathcal{B}(\tilde{C})(X) \approx \coprod_{m \geq 0} (P_m \times X^m)_{h\Sigma_m},$$

natural in CW-complexes X , in the category of simplicial objects in the homotopy category of spaces. In particular, for each $q \geq 0$ there are natural weak equivalences

$$\tilde{C}^{\circ(q+2)}(X) \approx \coprod_{m \geq 0} ((P_m)_q \times X^m)_{h\Sigma_m}.$$

Observe that there is a coproduct decomposition

$$\mathcal{B}(\tilde{C}) \approx \coprod_{m \geq 1} \mathcal{B}(\tilde{C})\langle m \rangle$$

in the category of simplicial functors, so that there are natural weak equivalences

$$\mathcal{B}(\tilde{C})\langle m \rangle(X) \approx (P_m \times X^m)_{h\Sigma_m}.$$

7.13. A fundamental observation. Our approach to proving that Δ is a Koszul ring comes from the following observation: if we apply E -homology to $\mathcal{B}(\tilde{C})(X)$ in each simplicial degree, and then “linearize” with respect to X , and set $X = *$, then what we obtain is the bar complex $\mathcal{B}(\Delta)$ for the ring Δ . This linearization, in turn, turns out to be exactly the transitive homology of the partition complex.

7.14. Proposition. *For $m = p^k$, there is an isomorphism of simplicial abelian groups*

$$\mathcal{B}(\Delta)[k] \rightarrow Q(P_m).$$

Furthermore, this isomorphism carries the subobjects $\hat{\mathcal{B}}(\Delta)[k]$, $\check{\mathcal{B}}(\Delta)[k]$, and $\ddot{\mathcal{B}}(\Delta)[k]$ isomorphically to the subobjects $Q(\hat{P}_m)$, $Q(\check{P}_m)$, and $Q(\ddot{P}_m)$.

Note that if m is not a power of p , then $\tilde{Q}(\overline{P}_m) = 0$. We give the proof of (7.14) below.

7.15. Corollary. *For $m = p^k$, $k \geq 0$, there is an isomorphism of simplicial abelian groups*

$$\overline{\mathcal{B}}(\Delta)[k] \approx \tilde{Q}(\overline{P}_m).$$

Proof. Immediate using (7.14) and (7.9). \square

Now we relate the simplicial object $\mathcal{B}(\mathcal{C})(X)$ with the algebraic functor \mathbb{T} described in §3.2.

7.16. Proposition. *If X is a finite Σ_m -set, $m = p^k$, then there is an isomorphism*

$$\mathcal{B}(\mathbb{T})\langle m \rangle(E_*X) \xrightarrow{\alpha} E_*^\wedge \mathcal{B}(\mathcal{C})\langle m \rangle(X)$$

of simplicial E_ -modules.*

Proof. This amounts to the fact that

$$(\mathbb{T}^{\circ q})\langle m \rangle(E_*X) \approx E_*^\wedge(O^{\circ q}(\underline{m}) \times X^m)_{h\Sigma_m}.$$

\square

Proof of (7.14). By (7.16) and (7.12), there are isomorphisms

$$\mathcal{B}(\tilde{\mathbb{T}})\langle m \rangle(E_*X) \approx E_*^\wedge \mathcal{B}(\tilde{C})\langle m \rangle(X) \approx E_*^\wedge(P_m \times X^m)_{h\Sigma_m}$$

of simplicial E_* -modules, natural in Σ_m -sets X . Applying (7.7), we see that

$$\mathcal{L}_{\mathcal{B}(\tilde{\mathbb{T}})\langle m \rangle}(\Sigma_+^\infty X) \approx Q(P_m \times X^m).$$

The result follows from the isomorphism

$$\mathcal{L}_{\mathcal{B}(\mathbb{T})}(M) \approx \mathcal{B}(\Delta) \otimes_{E_0} M$$

of (5.8).

To see that each of these isomorphisms induces isomorphisms of the relevant subobjects

$$\hat{\mathcal{B}}(\Delta)[k] \approx Q(\hat{P}_m), \quad \check{\mathcal{B}}(\Delta)[k] \approx Q(\check{P}_m), \quad \ddot{\mathcal{B}}(\Delta)[k] \approx Q(\ddot{P}_m),$$

we recall from (5.9) that the summands $\Delta[k_{q+1}] \otimes_{E_0} \cdots \otimes_{E_0} \Delta[k_0]$ of $\Delta^{\otimes(q+2)}$ come from the linearization of the pure part $\mathbb{T}\langle m_{q+1} \rangle \circ \cdots \circ \mathbb{T}\langle m_0 \rangle$ of $\tilde{\mathbb{T}}^{\circ(q+2)}$, where $m_i = p^{k_i}$. Tracing this through the isomorphisms, we see that these summands in the bar complex come from the transitive E -homology of a subset of $P_{m,q}$ consisting of chains $[E_0 \not\leq \cdots \not\leq E_q]$ of pure

partitions with $\text{mesh}(E_k) = m_0 \cdots m_k$. Such a summand will lie in $\hat{\mathcal{B}}(\Delta)[k]$ (resp. $\check{\mathcal{B}}(\Delta)[k]$) if and only if $m_0 > 1$ (resp. $m_{q+1} > 1$) if and only if the chains lie in $\hat{P}_{m,q}$ (resp. $\check{P}_{m,q}$). \square

8. RATIONAL CALCULATIONS AND REDUCTION TO THE UNIFORM COMPLEX

According to §4.4 and (7.15), to show that Δ is Koszul, we must show that the homology of the simplicial abelian group $\tilde{Q}(\overline{P}_m)$ is concentrated in degree k for $m = p^k$. To do this, we will effectively produce a chain homotopy equivalence between $N\tilde{Q}(\overline{P}_m)$ (the normalized chain complex of the simplicial abelian group), and $N\tilde{Q}(\overline{U}_m)$, where U_m is the uniform complex of §6.9. Then in the subsequent section, we will compute the homology of $\tilde{Q}(\overline{U}_m)$.

8.1. Rational calculations. We use the following homology version of the Hopkins-Kuhn-Ravenel character theorem. Given a finite group G , let $\mathcal{A}(G)$ denote the category whose objects are abelian subgroups A of G , and whose morphisms are maps of G -sets $G/A \rightarrow G/B$. There is an evident natural map

$$\alpha: \int^{A \in \mathcal{A}(G)} E_*^\wedge(BA \times X^A) \rightarrow E_*^\wedge X_{hG},$$

induced by the G maps $G/H \times X^A \rightarrow X$, where “ $\int^{C \in \mathcal{C}} F(C, C)$ ” denotes the coend of a bifunctor $F: \mathcal{C} \times \mathcal{C}^{\text{op}} \rightarrow \mathcal{D}$, i.e., the cokernel of the pair

$$\bigoplus_{C' \rightarrow C \in \mathcal{C}} F(C', C) \rightrightarrows \bigoplus_{C \in \mathcal{C}} F(C, C).$$

8.2. Proposition. *If X is a finite G CW-complex, then α becomes an isomorphism after inverting the order of G .*

Proof. This is a straightforward consequence of Theorem A of [HKR00]. \square

Now suppose that $G = \Sigma_m$, with $m = p^k$.

8.3. Proposition. *For finite G CW-complexes X , there is a natural isomorphism*

$$\bigoplus_{[A]} p^{-1} (F_A \otimes_{E_*} E_*^\wedge(X^A))_{\text{Aut}(A)} \approx p^{-1} Q(X),$$

where the sum runs over isomorphism classes of abelian groups of order m ,

$$F_A \stackrel{\text{def}}{=} \text{Cok} \left[\bigoplus_{A' \subsetneq A} E_*^\wedge(BA') \rightarrow E_*^\wedge(BA) \right],$$

where the sum is taken over proper subgroups of A .

Proof. The coend construction is natural in X and commutes with colimits. Thus, we see that after inverting p , $Q(X)$ must become isomorphic to $\int^{A \in \mathcal{A}(\Sigma_m)} K$, where

$$K(A', A) = \text{Cok} [E_*^\wedge(BA' \times (X \times S)^A) \rightarrow E_*^\wedge(BA' \times X^A)].$$

If A is not a transitive subgroup, then S^A is non-empty and so $K(A', A) = 0$. If A is transitive, we have $K(A', A) \approx E_*^\wedge(BA') \otimes_{E_*} E_*^\wedge(X^A)$. The result now follows using (6.4). \square

8.4. Corollary. *Let X be a finite Σ_m CW-complex such that $X^A = \emptyset$ for all transitive abelian subgroups A . Then $p^{-1}Q(X) = 0$.*

Finally, this is a convenient place to record the following.

8.5. Proposition. *For all $m, q \geq 0$, the objects $\tilde{Q}(P_m)_q$ are free E_* -modules.*

Proof. As already shown in (7.15)

$$\tilde{Q}((P_m)_q) \approx \mathcal{B}_q(\Delta)[k],$$

and we know that the $\Delta[k]$ are free as right E_* -modules, by (3.16). \square

8.6. Reduction to the uniform complex. Let (C_*, d) be a chain complex, and let $k \in \mathbb{Z}$. A k -**contraction** is a collection of functions $h = h_i: C_i \rightarrow C_{i+1}$, such that the function $f_i = \text{id} - (dh_i + h_{i-1}d): C_i \rightarrow C_i$ is equal to 0 if $i \neq k$. Thus, a k -contraction is a chain homotopy between the identity map and a chain map f which is non-zero only in dimension k . Thus, the existence of a k -contraction implies that $H_i C_* = 0$ for $i \neq k$.

The key reduction step is an application of the following lemma.

8.7. Lemma. *Let D_* and E_* be chain complexes of abelian groups, and let C_* be a subcomplex of D_* . We suppose the following.*

- (a) *There are chain maps $i: E_* \rightarrow D_*$ and $j: D_* \rightarrow E_*$ such that $\text{id} = ji: E_* \rightarrow E_*$.*
- (b) *For all $q \in \mathbb{Z}$, E_q is torsion free.*
- (c) *For all $q \in \mathbb{Z}$, C_q is a summand of D_q , so that there is a direct sum decomposition $D_q \approx C_q \oplus B_q$ with $d(C_q) \subseteq C_{q-1}$.*
- (d) *There exists an integer N such that $N B_q = 0$ for all q .*

Then if the chain complex C_ admits a k -contraction, then so does E_* .*

Proof. Let h be the hypothesized k -contraction of C_* , consisting of homomorphisms $h = h_q: C_q \rightarrow C_{q+1}$. Let $f = \text{id} - (dh + hd): C_* \rightarrow C_*$, so that $f_q = 0$ for $q \neq k$.

Define homomorphisms $h'_q: D_q \rightarrow D_{q+1}$ by “extending h by zero”. Thus, $h'_q|_{C_q} = h_q$, while $h'_q|_{B_q} = 0$. Let $f' = \text{id} - (dh' + h'd): D_* \rightarrow D_*$. It follows that $f'_q|_{C_q} = f_q$. Since $N f'_q|_{B_q} = 0$ by (d), it follows that $N f'_q = 0$ for $q \neq k$.

Now let $H = H_q = j h'_q i: E_q \rightarrow E_{q+1}$, and let

$$F = j f' i = ji - (j d h' i + j h' d i) = \text{id} - (dH + H d): E_* \rightarrow E_*.$$

Then $N F_q = j(N f'_q) i = 0$ for $q \neq k$, and thus $F_q = 0$ for $q \neq k$ since E_* is torsion free. It follows that H is a k -contraction on E_* . \square

We suppose that $m = p^k$, and that $P = P_m$, $U = U_m$, etc.

8.8. Proposition. *Suppose that $N\tilde{Q}(\overline{U})$ admits a k -contraction. Then so does $N\tilde{Q}(\overline{P})$.*

Proof. We show that the hypotheses of the lemma are satisfied, with $C_* = N\tilde{Q}(\overline{U})$, $D_* = N\tilde{Q}(\overline{P} \wedge CF_+)$ and $E_* = N\tilde{Q}(\overline{P})$.

Since $CF \approx \Sigma_m / \Sigma_p \wr \cdots \wr \Sigma_p$, we have by (7.5) maps of simplicial abelian groups $i: \tilde{Q}(\overline{P}) \rightarrow \tilde{Q}(\overline{P} \wedge CF_+)$ and $j: \tilde{Q}(\overline{P} \wedge CF_+) \rightarrow \tilde{Q}(\overline{P})$ such that $ji = \text{id}$; after passing to normalized chains, this is (a).

We have observed in (8.5) that the groups $N\tilde{Q}(\overline{P})_q$ are free modules, which is (b).

For each q , the essential q -simplices of $P \times CF$ can be decomposed as $T_q \coprod Y_q$, where T_q are the essential q -simplices contained in U , and Y_q are the essential q -simplices not in U . Thus

$$D_q = N\tilde{Q}(\overline{P} \wedge CF_+)_q \approx Q(T_q) \oplus Q(Y_q) \approx N\tilde{Q}(\overline{U})_q \oplus Q(Y_q),$$

and the boundary map for D_q restricts to the one on $N\tilde{Q}(\overline{U})_q$. Thus, taking $C_q = N\tilde{Q}(\overline{U})_q$ and $B_q = NQ(Y_q)$, we have (c).

Note that since $P^A \times CF^A = U^A$, we have that $Y_q^A = \emptyset$ for all transitive abelian subgroups A , and thus $p^{-1}Q(Y_q) = 0$ by (8.4). Since the sets Y_q are finite, and are empty for $q > m$, we can choose a positive integer N such that $p^N Q(Y_q) = 0$ for all q . Thus, in each degree q , the group $N\tilde{Q}(\overline{P} \wedge CF_+)_q$ has the form

$$N\tilde{Q}(\overline{U})_q \oplus Q(Y_q),$$

where $p^N Q(Y_q) = 0$, which is (d). \square

9. THE SHELLABILITY ARGUMENT

In this section, we complete the proof that Δ is Koszul, by proving that $\tilde{Q}(\overline{U}_m)$ admits a k -contraction for $m = p^k$. This will imply that $\tilde{Q}(\overline{P}_m)$ admits a k -contraction by (8.8), and thus so does $\tilde{B}(\Delta)[k]$ by (7.15).

The argument of this section is a generalization of the argument of Solomon and Tits [Sol69] on the homotopy type of the Tits building of a BN-pair, as realized in the case of the group $G = GL(k, \mathbb{F}_p)$ with its usual BN-structure. Here is a story which explains how this generalization is achieved, step by step.

- (1) Solomon and Tits show that the Tits building for G , which in our language is \ddot{P}_m^A with $A \approx (\mathbb{Z}/p)^k \subset \Sigma_m$ for $m = p^k$, is homotopy equivalent to a wedge of $p^{\binom{k}{2}}$ copies of the $(k-2)$ -sphere, by showing that the simplicial complex $X = \ddot{P}_k^A$ is “shellable”. That is, X is constructed by attaching $(k-2)$ -simplices in a certain order, so that each $(k-2)$ -simplex σ is attached along a subcomplex which is union of some of its codimension 1-faces. If we attach along a subcomplex which is not all of $\partial\sigma$, this subcomplex is contractible, and attaching σ does not change the homotopy type; if we attach along all of $\partial\sigma$, the attachment turns out to be homotopy equivalent to a one-point union with a $(k-2)$ -sphere.
- (2) Their argument is easily transformed into a proof that the pointed simplicial set \overline{P}_m^A is homotopy equivalent to a wedge of $p^{\binom{k}{2}}$ copies of the k -sphere; the argument still has a “shellability” character, though now we are attaching k -simplices one at a time.
- (3) Following an observation of Mitchell and Kuhn [KM86], we observe that this argument really gives, in the case $A \approx (\mathbb{Z}/p)^k$, a B -equivariant homotopy equivalence to $S^k \wedge (B/D)_+$, where B is the group of upper triangular matrices in G , and D the group of diagonal matrices in G (because the simplices can be glued on “one B -orbit at a time”). In our language, we can express this by saying that $\overline{U}_k^A = \overline{P}_m^A \wedge (G/B)_+$ is G -homotopy equivalent to $S^k \wedge (G/L)_+$, where L are the lower triangular matrices in G .
- (4) This “shellability” argument applies *without change* to arbitrary transitive abelian subgroups A of Σ_m . If A is not isomorphic to an elementary abelian p group, the

shellability argument shows that \overline{U}_m^A is $\text{Aut}(A)$ -equivariantly *contractible*. (When carrying out the shellability argument for such A , simplices σ are always attached along a *proper* subcomplex of $\partial\sigma$.)

- (5) Recall that $\overline{U}_m = \bigcup \overline{U}_m^A$. The homotopy equivalences constructed for each \overline{U}_m^A “agree with each other on overlaps”, since they are all constructed in the same way. Using this, \overline{U}_m can be shown to have a Σ_m -equivariant homotopy equivalence to $S^k \wedge (\Sigma_m/\Sigma_p^k)_+$.

We will not trace out all these steps, but rather carry out the last step directly. Thus, we will prove the following.

9.1. Proposition. *Let $m = p^k$. If \tilde{Q} is any functor from pointed Σ_m -sets to abelian groups which takes sums to sums, then the normalized chain complex $N\tilde{Q}(U_m)$ admits a k -contraction. In particular, this is the case when \tilde{Q} is transitive E -homology.*

The rest of the section is dedicated to the proof of (9.1).

9.2. The poset of subsets. Fix $k \geq 1$. Let $K = K_k$ denote the poset of subsets of the set $\underline{k} = \{1, \dots, k\}$, ordered by inclusion. Thus, a q -simplex of K corresponds to a chain $[S_0 \subseteq \dots \subseteq S_q]$ of subsets of \underline{k} . As in §6, we let $K^\diamond \subseteq K$ denote the sub-simplicial set consisting of chains such that either $S_0 \neq \emptyset$ or $S_q \neq \underline{k}$, and set $\overline{K} = K/K^\diamond$. An **essential** q -simplex of K is a non-degenerate simplex not contained in K^\diamond .

9.3. The map $\chi: U \rightarrow K$. We will define a map $\chi: U \rightarrow K$ of simplicial sets. First, suppose that A is a transitive abelian subgroup of Σ_m , where $m = p^k$, so that we can identify simplices in P^A , and elements of CF^A , with chains of subgroups of A .

Fix a complete chain $[V]$ in CF^A . We define a function $\chi_{[V]}^A$ from the set of subgroups of A to the set of subsets of \underline{k} , by

$$\chi_{[V]}^A(W) \stackrel{\text{def}}{=} \{j \in \underline{k} \mid \text{length}(V_j \cap W) - \text{length}(V_{j-1} \cap W) = 1\}.$$

That is, $\chi_{[V]}^A(W)$ is the set of indices at which there is a non-trivial filtration quotient in the filtration defined by $0 = V_0 \cap W \subseteq V_1 \cap W \subseteq \dots \subseteq V_k \cap W = W$.

Observe that if $W \subseteq W'$, then $\chi_{[V]}^A(W) \subseteq \chi_{[V]}^A(W')$ (since $(V_j \cap W)/(V_{j-1} \cap W) \rightarrow (V_j \cap W')/(V_{j-1} \cap W')$ is a monomorphism). Therefore, we obtain, for each $[V] \in CF^A$ a map of simplicial sets $\chi^A: U^A = P^A \times CF^A \rightarrow K$ by

$$\chi^A([W], [V]) = \left[\chi_{[V]}^A(W_0) \subseteq \chi_{[V]}^A(W_1) \subseteq \dots \subseteq \chi_{[V]}^A(W_q) \right].$$

Suppose that E is a pure partition (§6.8) of \underline{m} . We write $p^{\text{mesh}(E)}$ for the size of the equivalence classes of E . If A is a transitive abelian subgroup of Σ_m , and $V \subseteq A$ is a subgroup, then the V -orbit relation E_V is pure, with $\text{mesh}(E_V) = \text{length}(V)$.

If E_1 and E_2 are two equivalence relations on a set \underline{m} , we write $E_1 \cap E_2$ for the equivalence relation which is the common refinement of E_1 and E_2 ; i.e., $x \sim_{E_1 \cap E_2} y$ if and only if $x \sim_{E_i} y$ for $i = 1, 2$.

9.4. Proposition. *There is a unique map $\chi: U \rightarrow K$ of simplicial sets which extends the maps $\chi^A: U \rightarrow K$. The function χ is invariant with respect to the Σ_m -action on U . The*

preimage of a non-degenerate simplex of K under χ is a collection of non-degenerate simplices of U . The preimage of K^\diamond under χ is U^\diamond .

Proof. Given a q -simplex $\sigma = ([F], [E])$ of $U = P \times CF$, we will define $\chi(\sigma) = [S]$, where

$$S_i = \{ j \in \underline{k} \mid \text{mesh}(F_i \cap E_j) - \text{mesh}(F_i \cap E_{j-1}) = 1 \}.$$

It will soon be apparent that this is actually well-defined.

Suppose A is any transitive abelian subgroup of Σ_m which fixes the simplex σ of U . Then we have that $F_i = E_{W_i}$ and $E_j = E_{V_j}$ for some subgroups V_j, W_i of A , and we may write $\chi^A(\sigma) = [T]$, where $T_i = \{ j \in \underline{k} \mid \text{length}(W_i \cap V_j) - \text{length}(W_i \cap V_{j-1}) = 1 \}$.

It is clear that for subgroups W, W' of A , we have that $E_{W \cap W'} = E_W \cap E_{W'}$, and that the equivalence relation E_W satisfies $\text{mesh}(E_W) = \text{length}(W)$. Therefore, we have that $T_i = S_i$ for all $i = 0, \dots, q$, showing that $\chi(\sigma) = \chi^A(\sigma)$ for all A which fix σ , and thus incidentally showing that $\chi: U \rightarrow K$ is well-defined.

The Σ_m -invariance of χ is clear from the definition. The statements about preimages follow from the fact that if $W \subseteq W'$ are subgroups of A such that $\chi_{[V]}(W) = \chi_{[V]}(W')$, then $W = W'$. \square

9.5. Disorder filtration. We will now define a filtration $K^\diamond = F_{-1}K \subseteq F_0K \subseteq F_1K \subseteq \dots \subseteq F_{\binom{k}{2}}K = K$ of the simplicial set K , which we will use to define a Σ_m -equivariant filtration $U^\diamond = F_{-1}U \subseteq F_0U \subseteq F_1U \subseteq \dots \subseteq F_{\binom{k}{2}}U = U$ of U , by $F_jU \stackrel{\text{def}}{=} \chi^{-1}(F_jK)$.

Given a q -simplex $\sigma = [S] = [S_0 \subseteq \dots \subseteq S_q]$ of K , let $D(\sigma)$ denote the set of ordered pairs (a, b) with $1 \leq b < a \leq k$, such that there exists r with $1 \leq r \leq q$ such that $a \in S_r$ and $b \notin S_r$. (That is, a is “born before” b in the increasing sequence of sets $S_0 \subseteq \dots \subseteq S_q$, yet $a > b$.) Observe that $D(\tau) \subseteq D(\sigma)$ whenever τ is a face of σ .

Define $d(\sigma)$ to be the following integer:

- (1) if $\sigma \in K^\diamond$, set $d(\sigma) = -1$;
- (2) otherwise, $d(\sigma) = \#D(\sigma)$.

We call $d(\sigma)$ the **disorder** of the simplex. The maximal possible disorder is $\binom{k}{2}$.

If τ is a face of a simplex σ , then $D(\tau) \subseteq D(\sigma)$, so $d(\tau) \leq d(\sigma)$. Thus the disorder function defines an increasing filtration $\{F_rK\}$ of K , so that F_rK consists of the union of K^\diamond with all essential simplices σ with $d(\sigma) \leq r$. Thus $F_{-1}K = K^\diamond$ and $F_{\binom{k}{2}}K = K$. We write $F_r\overline{K}$ for the induced filtration on the quotient $\overline{K} = K/K^\diamond$.

Similarly, we can regard d as a function on the simplices of U , by $d(\sigma) = d(\chi(\sigma))$, so that the disorder function defines an increasing filtration $\{F_rU\}$ of U . Thus $F_{-1}U = U^\diamond$ and $F_{\binom{k}{2}}U = U$. We write $F_r\overline{U}$ for the induced filtration on the quotient $\overline{U} = U/U^\diamond$.

9.6. The subcomplex Λ_σ . Given an essential k -simplex σ of K of disorder $d(\sigma) = j$, let $\Lambda_\sigma \subseteq \Delta[k]$ be the subcomplex defined by the pullback square

$$\begin{array}{ccc} \Lambda_\sigma & \longrightarrow & F_{j-1}K \\ \downarrow & & \downarrow \\ \Delta[k] & \xrightarrow{\sigma} & F_jK \end{array}$$

In the following, it is convenient to identify a q -simplex σ of K with the subcomplex which is the image of the characteristic map $\sigma: \Delta[q] \rightarrow K$.

9.7. Proposition. *Let σ be an essential k -simplex of K . Then the subcomplex $\Lambda_\sigma \subseteq \partial\Delta[k]$ is a non-empty union of codimension 1 faces of the k -simplex; furthermore, $\Lambda_\sigma = \Delta[k]$ if and only if $d(\sigma) = \binom{k}{2}$.*

Proof. Suppose that $d(\sigma) = r$, and write $\sigma = [S] = [S_0 \subseteq \cdots \subseteq S_k]$. The intersection of σ with $F_{-1}K$ consists of the union of the two faces obtained by deleting $S_0 = \emptyset$ or $S_k = \underline{k}$ from the chain of subsets defining σ , and thus $\sigma \cap F_{r-1}K$ is certainly non-empty.

Now suppose that τ is a face of σ , with $d(\tau) \geq 0$. Then τ is necessarily the intersection of a collection of codimension 1-faces $\sigma_{i_1}, \dots, \sigma_{i_t}$ of σ , where σ_{i_j} is obtained by deleting S_{i_j} from the chain of subsets which defines σ . We have that $D(\tau) = \bigcap_{j=1}^t D(\sigma_{i_j})$. If $d(\sigma_{i_j}) = r$ for all j , then $D(\sigma_{i_j}) = D(\sigma)$ for all j , and so $D(\tau) = D(\sigma)$, whence $d(\tau) = r$. That is, if $d(\tau) < r$, then τ must be contained in some codimension 1 face σ_{i_j} with $d(\sigma_{i_j}) < r$. Thus $\sigma \cap F_{r-1}K$ is the union of all codimension 1 faces σ_i with $d(\sigma_i) < r$.

Every codimension 1 face σ_j of σ has $d(\sigma_j) < \binom{k}{2}$. Therefore, $d(\sigma) = \binom{k}{2}$ implies $\sigma \cap F_{\binom{k}{2}-1} = \partial\sigma$, i.e., that $\Lambda_\sigma = \partial\Delta[k]$.

Now write $S_i - S_{i-1} = \{s_i\}$, and let σ_j be the codimension 1 face of σ obtained by omitting S_j , where $0 < j < k$. If $d(\sigma_j) < d(\sigma)$, this can only be because $s_{j+1} < s_j$, since the sets $D(\sigma)$ and $D(\sigma_j)$ can differ by the one element (s_j, s_{j+1}) . Thus, if $r = d(\sigma)$ and $\partial\sigma = \sigma \cap F_{r-1}K$, we must have $s_{j+1} < s_j$ for all $j \in \{1, \dots, k-1\}$, whence $s_j = k - j + 1$, and thus σ is the unique simplex with maximal disorder $\binom{k}{2}$. \square

9.8. The disorder filtration of U , and the proof of (9.1). Let S_σ denote the set of essential k -simplices τ in U such that $\chi(\tau) = \sigma$, and let $g_\sigma: \Delta[k] \times S_\sigma \rightarrow U$ denote the tautological map. Note that Σ_m acts on the set S_σ , and g_σ is Σ_m -equivariant.

9.9. Proposition. *For each r such that $1 \leq r \leq \binom{k}{2}$, there exist pushout squares in the category of Σ_m -equivariant simplicial sets of the form*

$$\begin{array}{ccc} \coprod_{\sigma} \Lambda_\sigma \times S_\sigma & \longrightarrow & F_{r-1}U \\ \downarrow & & \downarrow \\ \coprod_{\sigma} \Delta[k] \times S_\sigma & \xrightarrow{(g_\sigma)} & F_r U, \end{array}$$

where the coproducts are taken over the set of essential k -simplices σ of K for which $d(\sigma) = r$.

Note that the square in (9.9) is in fact a *pullback* square; the assertion of the proposition is that it is also a pushout.

The proof of (9.9) will be the subject of the remainder of the section. First, we prove (9.1) using it.

9.10. Lemma. *Let $0 \rightarrow C_* \xrightarrow{f} D_* \xrightarrow{g} E_* \rightarrow 0$ be an exact sequence of chain complexes, such that each sequence $0 \rightarrow C_q \rightarrow D_q \rightarrow E_q \rightarrow 0$ splits.*

- (1) If E_* is chain homotopy equivalent to 0, then f is a chain homotopy equivalence.
- (2) If C_* is chain homotopy equivalent to 0, then g is a chain homotopy equivalence.

Proof. We prove (1); statement (2) is formally dual.

Choose splittings $D_q \approx C_q \oplus E_q$ in each degree. Let $H: E_q \rightarrow E_{q+1}$ be such that $dH + Hd = \text{id}_E$. Define $K: D_q \rightarrow D_{q+1}$ by $K(c, e) = (0, He)$, and let $u: D_* \rightarrow D_*$ be given by $u = \text{id} - (dK + Kd)$. Observe that u factors through a map $\bar{u}: D_* \rightarrow C_*$ such that $\bar{u}f = \text{id}$, whence \bar{u} is a homotopy inverse to f . \square

Proof of (9.1). For each $i = 1 \dots, \binom{k}{2}$, we have using (9.9) that $F_i U / F_{i-1} U \approx \bigvee_{\sigma} (\Delta[k] / \Lambda_{\sigma}) \wedge (S_{\sigma})_+$ as Σ_m -equivariant simplicial sets. Therefore,

$$\tilde{Q}(F_i U / F_{i-1} U) \approx \bigoplus_{\sigma} (\mathbb{Z} \Delta[k] / \mathbb{Z} \Lambda_{\sigma}) \otimes Q(S_{\sigma}),$$

as simplicial abelian groups, since Q preserves sums. If $i < \binom{k}{2}$, the normalized chain complexes $N(\mathbb{Z} \Delta[k] / \mathbb{Z} \Lambda_{\sigma})$ are chain homotopy equivalent to 0 (9.7), and from (9.10) we deduce that the projection $N\tilde{Q}(\bar{U}) \rightarrow N\tilde{Q}(U / F_{\binom{k}{2}-1} U)$ is a chain homotopy equivalence. Since $N\tilde{Q}(U / F_{\binom{k}{2}-1} U)$ is concentrated in the single dimension k , we see using (9.10) that $N\tilde{Q}(\bar{U})$ admits a k -contraction. \square

9.11. Best completions of simplices of K . The proof of (9.9) amounts to showing that every essential q -simplex in U of given disorder is a face of a *unique* essential k -simplex with the same disorder. We will first demonstrate this for simplices in K .

Suppose that $\sigma = [S]$ is an essential k -simplex of K . Let s_1, \dots, s_k be the ordering of elements of \underline{k} defined by $\{s_i\} = S_i - S_{i-1}$. Recall that the disorder $d(\sigma)$ of σ is the number of pairs (s_i, s_j) with $i < j$ and $s_i > s_j$. The only essential k -simplex with $d(\sigma) = 0$ corresponds to the sequence $(1, 2, \dots, k)$, while the unique essential k -simplex with $d(\sigma) = \binom{k}{2}$ corresponds to the sequence $(k, k-1, \dots, 1)$.

If $\tau = [T]$ is an essential q -simplex, with $q \leq k$, let s_1, \dots, s_k be the unique ordering of elements of \underline{k} such that

- (a) $s_i \in T_r$ and $s_j \notin T_r$ for some r implies $i < j$, and
- (b) if $s_i, s_j \in T_r - T_{r-1}$, then $s_i < s_j$ if and only if $i < j$.

That is, we obtain the s_i by listing the elements of T_1 in ascending order, followed by the elements of $T_2 - T_1$ in ascending order, and so on. Let $\sigma = [S]$ be the essential k -simplex defined by $S_i = \{s_1, \dots, s_i\}$. We call σ the **best completion** of τ . We record the following properties of the best completion.

9.12. Lemma. *Let $\sigma = [S]$ be the best completion of an essential q -simplex $\tau = [T]$. Then τ is a face of σ , and $D(\tau) = D(\sigma)$, whence $d(\tau) = d(\sigma)$.*

As above, write $\{s_i\} = S_i - S_{i-1}$. If c_i is the integer such that $s_i \in T_{c_i} - T_{c_i-1}$, then the following hold.

- (a) For all $i = 1, \dots, k$, either $c_i = c_{i-1}$ or $c_i = c_{i-1} + 1$ (where $c_0 = 0$).
- (b) If i is such that $c_i < c_{i+1}$, then $S_i = T_{c_i}$.
- (c) We have $S_i = S_{i-1} \cup (\underline{s_i} \cap T_{c_i}) = \bigcup_{j=1}^i (\underline{s_j} \cap T_{c_j})$ for all $i = 1, \dots, k$, where $\underline{s_i} = \{1, 2, \dots, s_i\}$.

Proof. It is clear from the construction that for given $j = 1, \dots, q$, there is a (unique) $b_i \in \{1, \dots, k\}$ such that $T_j = \{s_1, \dots, s_{b_j}\} = S_{b_j}$, whence σ has τ as a face. It is clear that $D(\tau) = D(\sigma)$.

Since τ is essential, $T_j - T_{j-1}$ is non-empty, giving (a). Property (b) follows. Property (c) is straightforward. \square

9.13. Proposition. *Let $\tau = [T]$ be an essential q -simplex of K . Then its best completion $\sigma = [S]$ is the unique essential k -simplex such that (i) τ is a face of σ and (ii) $d(\sigma) = d(\tau)$.*

Proof. We have already shown (9.12) that the best completion of τ satisfies the desired properties.

Consider any essential k -simplex $\sigma = [S]$ such that τ is face of σ and $d(\sigma) = d(\tau)$, which implies that $D(\sigma) = D(\tau)$. Let s_i be such that $\{s_i\} = S_i - S_{i-1}$. For $j = 1, \dots, q$, let b_j be the unique index such that $T_j = S_{b_j}$, whence $T_j - T_{j-1} = \{s_{b_{j-1}+1}, \dots, s_{b_j}\}$. Then the condition that $D(\sigma) = D(\tau)$ implies that $s_{b_{j-1}+1} < \dots < s_{b_j}$, and thus σ is uniquely determined, and is seen to be the best completion of τ . \square

9.14. More on filtration jumps. We fix a transitive abelian subgroup A of Σ_m , and a complete filtration $[V] \in CF^A$. We recall the function $\chi = \chi_{[V]}^A$ which associates to each subgroup W of A a subset of $\underline{k} = \{1, \dots, k\}$.

9.15. Proposition. *Fix a transitive abelian subgroup A and a complete chain $[V]$ in CF^A . Let W and W' be subgroups of A .*

- (1) *If $W \subseteq W'$, then $\chi(W) \subseteq \chi(W')$.*
- (2) *$\#\chi(W) = \text{length}(W)$, and more generally $\#(\underline{r} \cap \chi(W)) = \text{length}(V_r \cap W)$.*
- (3) *If $W \subseteq W'$ and $\chi(W) = \chi(W')$, then $W = W'$.*
- (4) *$\chi(W + W') \supseteq \chi(W) \cup \chi(W')$.*
- (5) *$\chi(W \cap W') \subseteq \chi(W) \cap \chi(W')$.*
- (6) *$\chi(V_r \cap W) = \underline{r} \cap \chi(W)$.*

Proof. Statement (1) follows from the fact that $(V_j \cap W)/(V_{j-1} \cap W) \rightarrow (V_j \cap W')/(V_{j-1} \cap W')$ is a monomorphism. Statement (2) is clear from the definition, and statement (3) is immediate from (1) and (2). Statements (4) and (5) follow from (1), since $W + W' \supseteq W$, $W' \supseteq W \cap W'$. For statement (6), note that (5) implies $\chi(V_r \cap W) \subseteq \chi(V_r) \cap \chi(W) = \underline{r} \cap \chi(W)$. Since $\#(\underline{r} \cap \chi(W)) = \text{length}(V_r \cap W)$ by (2), and also $\#\chi(V_r \cap W) = \text{length}(V_r \cap W)$ using (2), the equality follows. \square

9.16. Proposition. *Fix a transitive abelian subgroup A and a complete chain $[V]$. Suppose $W \subseteq T$ are subgroups of A , and that $r \in \{0, \dots, k\}$. Then*

$$\chi(W + V_r \cap T) = \chi(W) \cup (\underline{r} \cap \chi(T)).$$

Proof. We have

$$\chi(W + V_r \cap T) \supseteq \chi(W) \cup (\underline{r} \cap \chi(T))$$

by (9.15), (4) and (6). Thus, it will suffice to show that these two sets have the same size.

Since $W \subseteq T$ and $\chi(W) \subseteq \chi(T)$, we have $W \cap (V_r \cap T) = V_r \cap W$ and $\chi(W) \cap (\underline{x} \cap \chi(T)) = \underline{x} \cap \chi(W)$. Thus

$$\begin{aligned} \#\chi(W + V_r \cap T) &= \text{length}(W + V_r \cap T) && \text{by (9.15) (2)} \\ &= \text{length}(W) + \text{length}(V_r \cap T) - \text{length}(V_r \cap W) \\ &= \#\chi(W) + \#(\underline{x} \cap \chi(T)) - \#(\underline{x} \cap \chi(W)) && \text{by (9.15) (6)} \\ &= \#(\chi(W) \cup (\underline{x} \cap \chi(T))). \end{aligned}$$

□

9.17. Proposition. *Fix a transitive abelian subgroup A and a complete chain $[V]$. Suppose $W \subseteq W' \subseteq T$ are subgroups of A , and that $r \in \{0, \dots, k\}$. Then the following are equivalent.*

- (a) $W' = W + V_r \cap T$.
- (b) $\chi(W') = \chi(W) \cup (\underline{x} \cap \chi(T))$.

Proof. That (a) implies (b) is just (9.16).

To show that (b) implies (a), observe that

$$W' \subseteq W' + V_r \cap T \supseteq W + V_r \cap T.$$

Thus by (9.15) (1),

$$\chi(W') \subseteq \chi(W' + V_r \cap T) \supseteq \chi(W + V_r \cap T).$$

By (9.16) and (b) we have that

$$\chi(W' + V_r \cap T) = \chi(W') \cup (\underline{x} \cap \chi(T)) = \chi(W) \cup (\underline{x} \cap \chi(T)),$$

and, again by (9.16) and (b) we have

$$\chi(W + V_r \cap T) = \chi(W) \cup (\underline{x} \cap \chi(T)) = \chi(W'),$$

and thus $W' = W' + V_r \cap T = W + V_r \cap T$ by (9.15) (3). □

9.18. The disorder filtration of U^A . Recall that for a transitive abelian subgroup A of Σ_m , we have defined a map of simplicial sets $\chi^A: U^A \rightarrow K$.

Given an essential q -simplex $\tau = ([W], [V])$ of U^A , let $\bar{\tau} = [T] = \chi^A(\tau)$ be the corresponding essential q -simplex of K , and let $\bar{\sigma} = [S]$ be the “best completion” of $\bar{\tau}$. Define $\sigma = ([W'], [V])$ to be the k -simplex of U^A defined by

$$(9.19) \quad W'_i \stackrel{\text{def}}{=} V_{s_1} \cap W_{c_1} + \dots + V_{s_i} \cap W_{c_i},$$

where s_i and c_i are such that $\{s_i\} = S_i - S_{i-1}$, and $s_i \in T_{c_i} - T_{c_{i-1}}$. We call σ the **best completion** of τ . Note that we also have the formula

$$W'_i = W'_{i-1} + V_{s_i} \cap W_{c_i}$$

for $i = 1, \dots, k$.

9.20. Lemma. *Let τ be an essential q -simplex of U^A , and let σ be its best completion in U^A . Then σ is an essential k -simplex of U^A , the simplex τ is a face of σ , the simplex $\chi^A(\sigma)$ is the best completion of $\chi^A(\tau)$ in K , and $d(\sigma) = d(\tau)$.*

Proof. Let $\overline{\sigma} = [S]$ denote the best completion of $\overline{\tau} = [T] = \chi^A(\tau)$. First we show that $\chi(W'_i) = \chi_{[V]}^A(W'_i) = S_i$, by induction on i , starting with $\chi(W'_0) = \emptyset$. If $i \geq 0$, then $W'_i = W'_{i-1} + V_{s_i} \cap W_{c_i}$ and $W'_{i-1} \subseteq W'_i \subseteq W_{c_i}$, and so

$$\begin{aligned} \chi(W'_i) &= \chi(W'_{i-1}) \cup (\underline{s_i} \cap \chi(W_{c_i})) && \text{by (9.16),} \\ &= S_{i-1} \cup (\underline{s_i} \cap T_{c_i}) && \text{by induction on } i, \\ &= S_i. \end{aligned}$$

This implies that σ is essential, and that $\chi^A(\sigma) = \overline{\sigma}$, implying $d(\sigma) = d(\tau)$.

Finally, note that if $i \in \{0, \dots, k-1\}$ is such that $c_{i+1} > c_i$, we must have $S_i = T_{c_i}$ by (9.12), which together with $W'_i \subseteq W_{c_i}$ implies $W'_i = W_{c_i}$. Thus, τ is a face of σ . \square

9.21. Lemma. *Suppose τ is an essential q -simplex of U^A . Then the best completion σ of τ is the unique essential k -simplex σ of U^A such that (i) τ is a face of σ , and (ii) $d(\tau) = d(\sigma)$.*

Proof. We have already shown (9.12) that the best completion satisfies the conditions, so the only remaining part is the uniqueness statement. Thus, suppose that $\sigma = ([W'], [V])$ is an arbitrary k -simplex which has τ as a face and such that $d(\tau) = d(\sigma)$. It follows that $\chi^A(\sigma) = \overline{\sigma} = [S]$ is the best completion of $\chi^A(\tau) = \overline{\tau} = [T]$. We are done once we prove (by induction on i) that $W'_i = W'_{i-1} + V_{s_i} \cap W_{c_i}$, where $\{s_i\} = S_i - S_{i-1}$ and $s_i \in T_{c_i} - T_{c_i-1}$. Writing χ for $\chi_{[V]}^A$, we have

$$\chi(W'_i) = S_i = S_{i-1} \cup (\underline{s_i} \cap T_{c_i}) = \chi(W'_{i-1}) \cup (\underline{s_i} \cap \chi(W_{c_i})),$$

and thus by applying (9.17) to the sequence $W'_{i-1} \subseteq W'_i \subseteq W_{c_i}$, we see that $W'_i = W'_{i-1} + V_{s_i} \cap W_{c_i}$. \square

9.22. The disorder filtration of U . Recall that if E_1 and E_2 are two equivalence relations on a set \underline{m} , we write $E_1 \cap E_2$ for the equivalence relation which is the common refinement of E_1 and E_2 ; i.e., $x \sim_{E_1 \cap E_2} y$ if and only if $x \sim_{E_1} y$ and $x \sim_{E_2} y$. We also write $E_1 + E_2$ for the equivalence relation which is generated by E_1 and E_2 ; i.e., the finest equivalence relation E such that $x \sim_{E_1} y$ or $x \sim_{E_2} y$ implies $x \sim_E y$.

9.23. Lemma. *Suppose that A acts transitively on \underline{m} , with $m = p^k$. Then for all subgroups V, V' of A , we have*

- (1) $\text{mesh}(E_V) = \text{length}(V)$,
- (2) $E_{V \cap V'} = E_V \cap E_{V'}$,
- (3) $E_{V+V'} = E_V + E_{V'}$.

Proof. Straightforward, using the fact that A acts transitively and freely on \underline{m} . \square

9.24. Lemma. *Suppose τ is an essential q -simplex of U . Then there exists a unique k -simplex σ in U such that (i) τ is a face of σ , and (ii) $d(\tau) = d(\sigma)$.*

Proof. Note that if $\tau \in U^A$ for some transitive abelian A , then there exists a unique k -simplex $\sigma_A \in U^A$ with the given property, by (9.21). Furthermore, observe that since σ_A is the “best completion” of τ in U^A , it is described by formulas (9.19) which involve only knowledge of the simplex $\chi(\tau)$ of K , and the operations of sum and intersection of subgroups of A . By (9.23), each of these operations can be described entirely in terms of the equivalence relations

generated by such subgroups. Thus, if $\tau \in U^A \cap U^{A'}$ with respective best completions σ_A and $\sigma_{A'}$ in U^A and $U^{A'}$, we must have $\sigma_A = \sigma_{A'}$. \square

Finally, we can give the proof of (9.9).

Proof of (9.9). By (9.24), every essential simplex τ of U is the face of an essential k -simplex σ with $d(\tau) = d(\sigma)$ by (9.24)(1), and thus $F_j U$ is a quotient of the images of $F_{j-1} U$ and the $\Delta[k] \times S_\sigma$. The square is a pullback by (9.24)(2). A simplex in $F_j U - F_{j-1} U$ is the image of a *unique* simplex in $\coprod_\sigma \Delta[k] \times S_\sigma$, by the uniqueness part of (9.24)(1). It follows that the square is a pushout. \square

REFERENCES

- [And95] Matthew Ando, *Isogenies of formal group laws and power operations in the cohomology theories E_n* , Duke Math. J. **79** (1995), no. 2, 423–485.
- [AM99] Greg Arone and Mark Mahowald, *The Goodwillie tower of the identity functor and the unstable periodic homotopy of spheres*, Invent. Math. **135** (1999), no. 3, 743–788.
- [HKR00] Michael J. Hopkins, Nicholas J. Kuhn, and Douglas C. Ravenel, *Generalized group characters and complex oriented cohomology theories*, J. Amer. Math. Soc. **13** (2000), no. 3, 553–594 (electronic).
- [JM04] B. Johnson and R. McCarthy, *Deriving calculus with cotriples*, Trans. Amer. Math. Soc. **356** (2004), no. 2, 757–803 (electronic).
- [Kas95] Takuji Kashiwabara, *$K(2)$ -homology of some infinite loop spaces*, Math. Z. **218** (1995), no. 4, 503–518.
- [KM86] N. J. Kuhn and S. A. Mitchell, *The multiplicity of the Steinberg representation of $GL_n \mathbf{F}_q$ in the symmetric algebra*, Proc. Amer. Math. Soc. **96** (1986), no. 1, 1–6.
- [McC83] James E. McClure, *Dyer-Lashof operations in K -theory*, Bull. Amer. Math. Soc. (N.S.) **8** (1983), no. 1, 67–72.
- [PP05] Alexander Polishchuk and Leonid Positselski, *Quadratic algebras*, University Lecture Series, vol. 37, American Mathematical Society, Providence, RI, 2005.
- [Pri70] Stewart B. Priddy, *Koszul resolutions*, Trans. Amer. Math. Soc. **152** (1970), 39–60.
- [Rez09] Charles Rezk, *The congruence criterion for power operations in Morava E -theory*, Homology, Homotopy Appl. **11** (2009), no. 2, 327–379, available at [arXiv:0902.2499](https://arxiv.org/abs/0902.2499).
- [Rez12] ———, *Modular isogeny complexes* (2012), available at [arXiv:arXiv:1102.5022](https://arxiv.org/abs/1102.5022).
- [Sol69] Louis Solomon, *The Steinberg character of a finite group with BN -pair*, Theory of Finite Groups (Symposium, Harvard Univ., Cambridge, Mass., 1968), Benjamin, New York, 1969, pp. 213–221.
- [Str97] Neil P. Strickland, *Finite subgroups of formal groups*, J. Pure Appl. Algebra **121** (1997), no. 2, 161–208.
- [Str98] N. P. Strickland, *Morava E -theory of symmetric groups*, Topology **37** (1998), no. 4, 757–779, available at [arXiv:math/9801125](https://arxiv.org/abs/math/9801125).

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